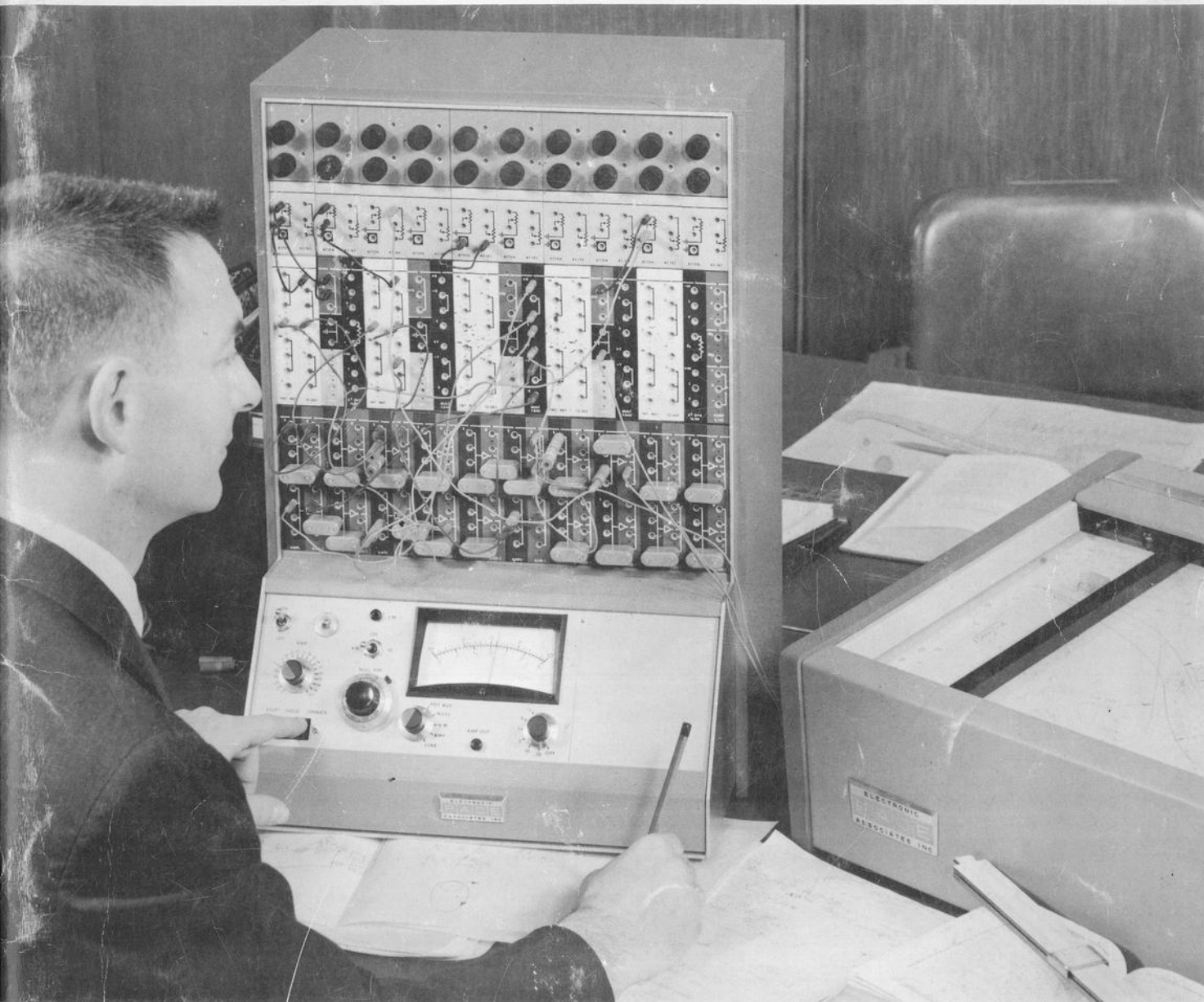# PACE® TR-10 *analog computer*

operator's handbook

EAI

ELECTRONIC ASSOCIATES, INC. *Long Branch, New Jersey*

PRICE $1.00 PER COPY.

PACE® TR-10

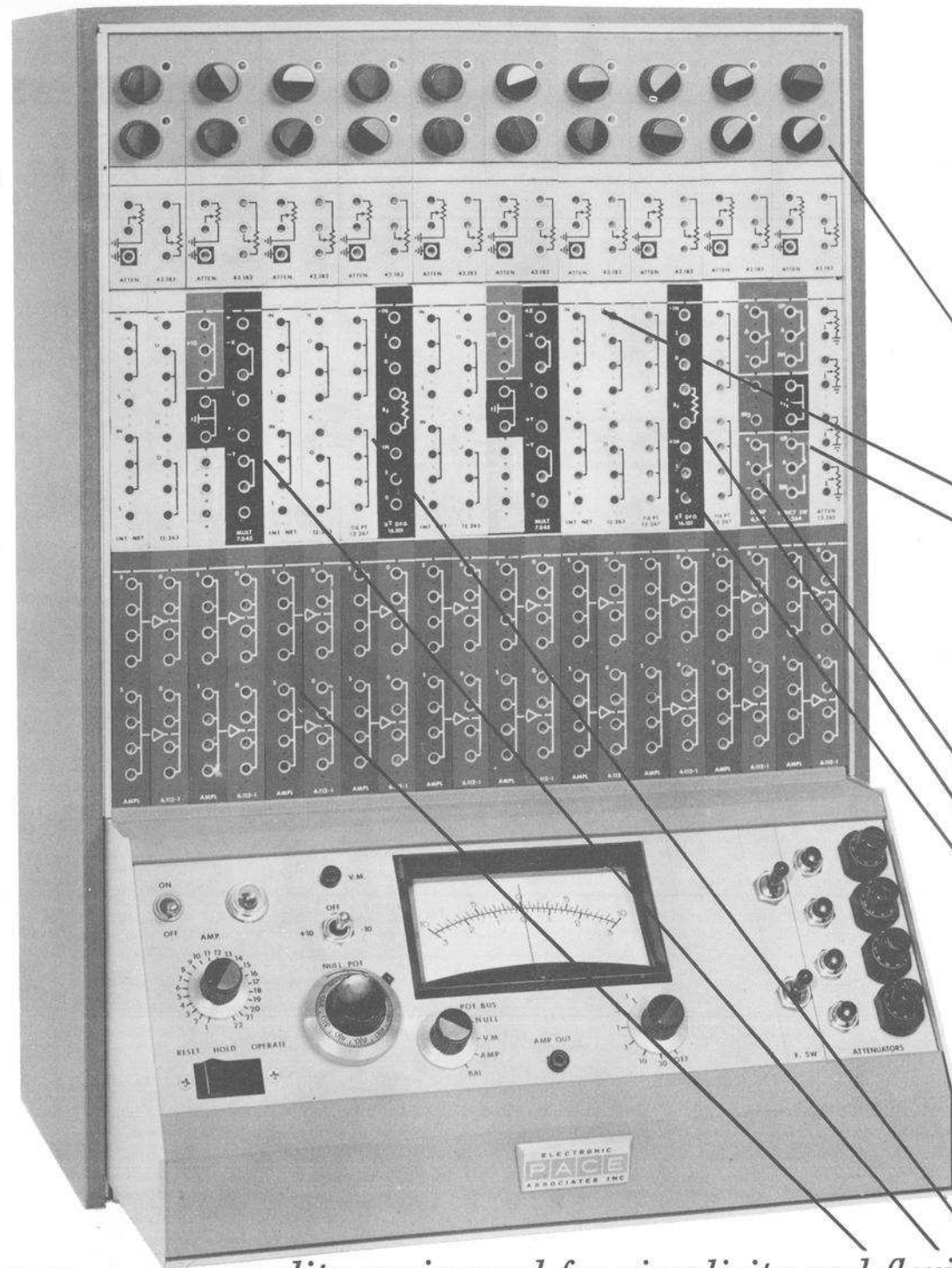*Transistorized Analog Computer*

operator's handbook

ATTENUATOR ROW

NON-LINEAR ROW

AMPLIFIER ROW

CONTROL PANEL

COEFFICIENT SETTING POTENTIOMETERS
for inserting equation coefficients or problem parameters into computer.

INTEGRATOR NETWORKS
enable operational amplifiers to perform integration.

FUNCTION SWITCHES
provide for manually interchanging components without reprogramming or repatching.

COMPARATORS
compare a variable input voltage to an arbitrary bias voltage and cause a switching operation to be performed.

FUNCTION GENERATORS
electronically generate analytic, as well as arbitrary functions of one variable.

REFERENCE PANEL
makes available accurate reference voltages required for problem solution.

TIE POINT PANELS
provide additional patch panel terminations for components inputs or outputs.
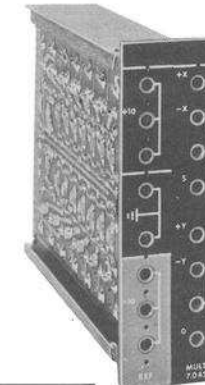
MULTIPLIERS
electronically multiply two variables of either sign.
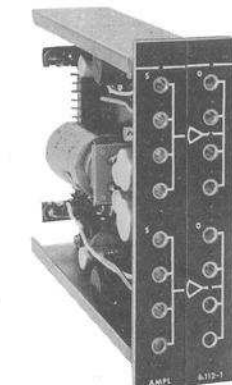
OPERATIONAL AMPLIFIERS
are high-gain, low-drift, chopper-stabilized DC amplifiers used for addition, subtraction, integration and inversion. With other components, they also perform a variety of non-linear operations.
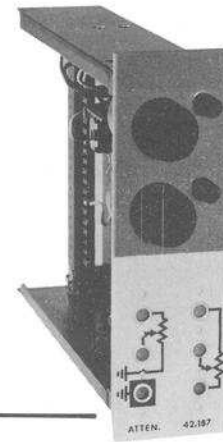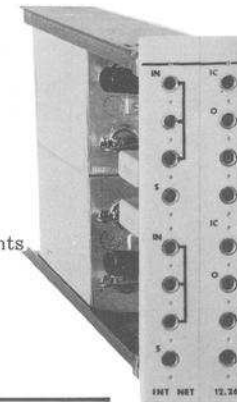
PACE® TR-10 *quality engineered for simplicity and flexibility*

- **Plug-in components** may be replaced easily and quickly for expansion or servicing.
- **Non-linear components** fit most non-linear row positions. Number of configurations is limited only by the number of components kept on hand.
- **Basic Computer** is pre-wired and can be expanded simply by plugging in desired components — no additional wiring necessary.
- **Draws less power than 60 watt bulb.** Operates from 115V, 60 cycle outlet.
  Bus bar power distribution eliminates complex cabling and simplifies maintenance.

- **Solid state design** — instant warm-up — no cooling problems.
- **Human engineered control panel** is inclined for easy, finger-tip control.
- **Push button potentiometer** readout system speeds set up — reduces errors.
- **Built-in null voltmeter** provides direct reading or precision null reading.
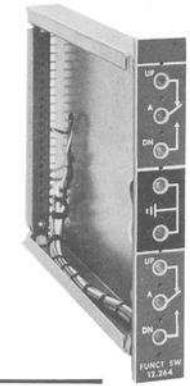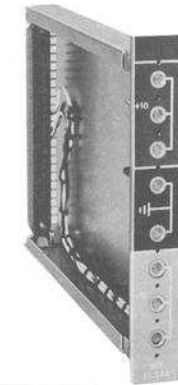- **Color coded** patching modules and accessories promote programming efficiency.

- Service Shelf (Type 51.039)
  Facilitates maintenance of any
  plug-in computing component
  under normal operating conditions.
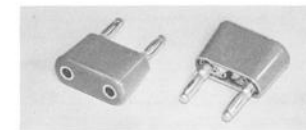
Feedback Resistor
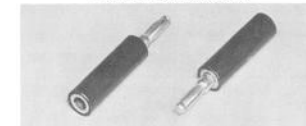
Input Resistor

Diode Unit
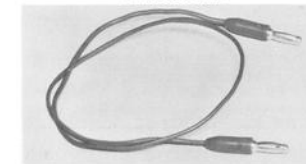
Patch Cord

- Feedback Resistor
  Type 646.010 — 10,000 ohms, ±0.1% wire-round resistor;
  supplied in a *BLUE* molded plug, is designed for patching
  between summing junction and output terminations of any
  operational amplifier.
- Input Resistors
  Type 646.006 — 10,000 ohms, ±0.1%., an *ORANGE* banded
  epoxy-encapsulated, wire-wound resistor — male end plugs
  into amplifier summing junction termination, female end
  accepts patch cord plug.
  Type 646.007 — 100,000 ohms, ±0.1%, a *YELLOW* banded,
  expoxy-encapsulated, wire-wound resistor — male end plugs
  into amplifier summing junction termination, female end
  accepts patch cord plug.
- Resistor Set (Type 5.134)
  Includes the following quantities of Input and Feedback
  Resistors described above:
  15 each Type 646.006 (see above)
  10 each Type 646.007 (see above)
  10 each Type 646.010 (see above)
- Diode Unit (Type 614.051)
 A *WHITE* banded, epoxy-encapsulated silicon diode for
  limiting the output of computing components or generating
  non-linear effects.
- Patch Cord Set (Type 5.133), includes the following:
  10 each Type 510.033-0, 6" long, Color — Black
  15 each Type 510.033-1, 12" long, Color — Brown
  10 each Type 510.033-2, 18" long, Color — Orange
  5 each Type 510.033-4, 30" long, Color — Blue
- Multiple Block (Type 542.605)
  Provides a six hole, off-the-patch-panel tie point for
  interconnecting patch cords or increasing the number of
  output holes of various computing components.

CONTENTS

BASIC PROGRAMMING PROCEDURES FOR
TRANSISTORIZED ANALOG COMPUTERS

INTRODUCTION

Many of the problems encountered in scientific or engineering endeavors require the solution of mathematical equations or sets of equations which in most cases are difficult and in other instances are, for all practical purposes, impossible to obtain by the classical approach to equation solution. The PACE TR-10 Analog Computer aids in overcoming this difficulty by providing the technical worker with a simple-to-use computer which permits the rapid solution of linear or non-linear mathematical equations.

Although the analog machine is correctly termed a computer, it does not perform its computations by numerical calculations as does the desk calculator or the digital computer. The analog computer performs mathematical operations on continuous variables instead of counting with digits. Thus the analog computer does not subtract 20 inches from 45 inches to obtain 25 inches but, rather, it subtracts 4 volts from 9 volts to obtain 5 volts. This answer the operator reads as 25 inches in accordance with his arbitrarily specified "scale factor" of 1 volt equals 5 inches. In the PACE TR-10, as is the case with the majority of modern analog computers, the continuous variables are d-c voltages.

The electronic analog computer makes possible the building of an electrical model of a physical system in which d-c voltages will behave with time in a similar way to the variables of interest in the actual system. If the vertical position of the center of gravity of an automobile oscillates with time during a disturbance, then the voltage representing the height of the center of gravity above the road surface will also oscillate; if the temperature of the coolant water at the exhaust port of the condenser rises exponentially to a steady value as a power plant is put into operation, then so will the voltage representing it on the computer. It can be said that the actual system and the electrical model are analogous in that the variables which demonstrate their characteristics are described by relations which are mathematically equivalent. The actual system has thus been "simulated" because of the similarity of operation of the electrical model and the actual system. These capabilities of analog computers are of great value in performing scientific research or engineering design calculations in that they give an insight into the relationship between the mathematical equations and the response of the physical system. Although the analog computer operates as a simulator, it performs basically as an equation solver since it performs mathematical operations which result in solutions of the equations used to represent the actual system.

Although the analog computer utilizes electronic components and electrical circuit characteristics in its operation, it is not essential that its user have an extensive knowledge of electrical circuits. The task of preparing on the computer the correct electrical model is simple, and the steps necessary for accomplishing this task once the mathematical description of the primary system is known will be detailed in this manual. Once the electrical model is completed, experiments can be performed cheaply, quickly and with great flexibility to predict the behavior of the primary physical system under many different conditions, and it is with this in mind that one builds the model. The intention here is to give the scientist or engineer using an analog computer for the first time a comprehensive introduction to its functions, its mathematical programming, and its operation to achieve usable experimental results.

The analog computer is basically a set of building blocks, each able to perform specific mathematical operations on direct voltages and capable of being easily interconnected one to another. By constructing an appropriately interconnected group of building blocks, an electrical model is produced in which the voltages at the outputs of the blocks obey the relations given in the mathematical description of a physical problem. As our interest is frequently in the dynamic behavior of physical systems, the mathematical

equations are usually differential equations having time as the independent variable. By applying appropriate initial conditions and forcing functions to the electrical model, its behavior is determined to be the same as that of the primary system. This behavior is viewed through output devices which produce traces of voltage values plotted one against another on an x-y plotter or against time on a strip-chart recorder. Steady values of voltage are viewed on a voltmeter which is an integral part of the computer and can be connected by a selector switch or by push buttons to the output terminations of many of the computing blocks.

In reading this manual, the reader ought to gain from the first section sufficient knowledge of the computer operation to enable him to interconnect computing units and to operate the computer without difficulty, from the second section, an understanding of simple programming procedures and computing techniques to allow him to construct a computer circuit diagram for any straightforward investigation, and, from the third section, an introductory view of more involved analog computer studies which will indicate to him the investigations possible with a larger complement of equipment. Needless to say, the full story cannot be told in a short manual. There are many lessons to be learned in the art of analog computing, and, as in every practical subject, experience is to be preferred over reading. However, the ideas and facts presented here ought to set the reader well along the best path to the goal of dynamic problem investigation by simple effective use of the modern analog computer.

I. THE COMPUTER

INTRODUCTION

The PACE TR-10 Analog Computer is a fully transistorized, general purpose analog computer. Solid state circuit elements are used throughout the computer to eliminate vacuum tubes, thereby achieving a compact design which requires very little power. It is able to operate stably and accurately in normal office surroundings. Reliable, with simplicity in functional design, it is easy to use and can be a powerful tool for the individual engineer in the rapid solution of scientific and engineering problems.

Consistent with our objective of programming the computer by interconnecting its components to solve mathematical equations, we will concentrate our attention on the computing components — the building blocks for our electrical model. These blocks are able to perform the following operations on variable d-c voltages — multiplication by a constant, algebraic summation, integration with respect to time, multiplication of two variables, generation of known functions of a variable, or combinations of these operations. Each component has input terminations and output terminations which are readily accessible at the front face of the computer for interconnection by plugs and patch cords.

Below the patching area lies the monitoring and control panel. This contains features which permit (a) switching the computer on and off, (b) controlling the operational mode of the computer, (c) setting the values of problem parameters to three-place accuracy, (d) reading out stationary values of problem variables, and (e) periodically adjusting the balance of the d-c amplifiers to ensure their accurate operation.

The front face of the computer is divided into three five-inch high rows of computing components and their corresponding interconnecting terminations. This will be referred to as the "patch panel". In the top row there are *attenuators* for multiplying voltages by positive constants less than unity. In the bottom row there are *high-gain d-c amplifiers*, uncommitted in their form of operation and, as we shall see later, capable of performing many tasks. In the middle row there is an assortment of components and terminations — *integrator networks* for use with the d-c amplifiers, *fixed and variable function generators, quarter-square multipliers, comparators*, and terminations for *additional control panel mounted attenuators, function switches, the reference voltages of ±10 volts, and ground potential.* These computing components will be the building blocks in forming any model and their characteristics and uses will be explained in sufficient detail to enable the reader to make use of them in his problems.

ATTENUATORS

Probably the simplest useful operation performed on the computer is that of multiplying a variable voltage by a positive constant whose value lies between zero and unity -- in electrical terminology, attenuation. A simple potentiometer is able to perform this task effectively to an accuracy of 0.1%. Thus, one is able to adjust problem equation coefficients and problem initial values by setting of the appropriate attenuator.

Four types of potentiometer modules are available, as shown in the accompanying *Figure 1.*

Attenuator 42.183. Ten-turn carbon potentiometers (5K ohms) with uncalibrated knobs. The upper potentiometer has one end grounded and thus has only two terminations immediately below the knobs and on the left side of the module. The lower potentiometer has both ends available and thus with the arm has three terminations at the lower right of the module. A ground connection is conveniently placed so that when, as is frequently required, one end of the attenuator needs to be grounded, a short patching cable can be used. The arm of each potentiometer constitutes the output of the device. It is terminated a second time for monitoring purposes at the upper right of the adjusting knob.
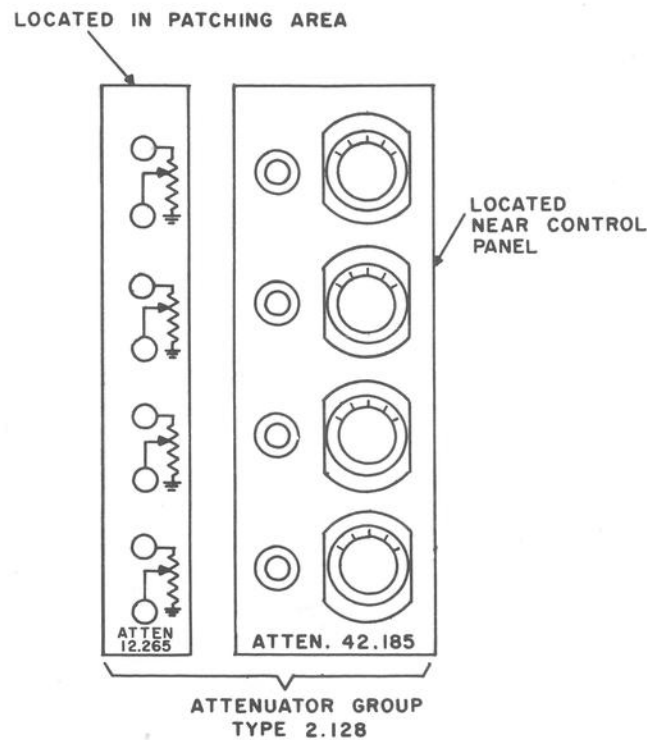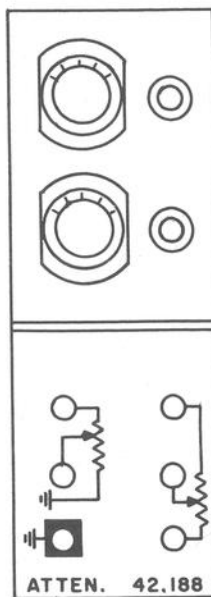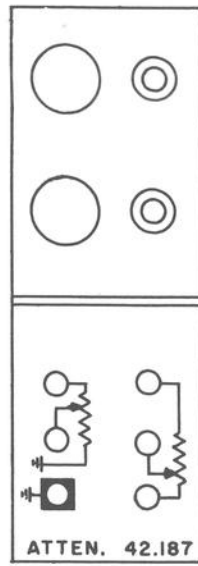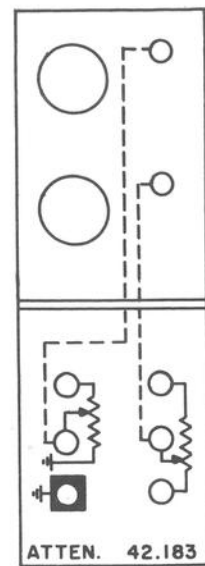
LOCATED IN PATCHING AREA

LOCATED NEAR CONTROL PANEL

ATTENUATOR GROUP
TYPE 2.128

Figure 1. Attenuator Modules

Attenuator 42.187. Same as for 43.183 except that in place of the second arm-termination a more convenient arrangement is available for adjusting the coefficient setting of the potentiometer. A push button is used to replace the patched signal voltage input by +10 volts and to connect the arm of the potentiometer to a *Pot Set Bus.* With the appropriate positioning of the Meter Mode Selector Switch this connects the output of the attenuator directly to the voltmeter for suitable adjustment (discussed later). No auxiliary patching is required to adjust this attenuator accurately as is required for 42.183.

Attenuator 42.188. Ten-turn, wire-wound potentiometers (5K ohms) with calibrated dials. Same as for 42.187 except for a change in the potentiometer used.

Attenuator 2.128. This group of four ten-turn, wire-wound potentiometers (5K ohms) with calibrated dials is positioned at the right of the control panel. The potentiometer terminations are at the right side of the middle row of the patch panel. All potentiometers are grounded, having simply one input and an output termination. Push buttons are available for ease in adjusting these attenuators and the intention is that they be used in a computer model to effect changes in the important design parameters of an investigation.

Whenever the coefficient setting of an attenuator is to be adjusted with accuracy on the computer, an electrical check of the setting must be used. A mechanical adjustment could be inaccurate for a number of reasons. For a fair degree of accuracy, applying +10 volts to the input and reading the output voltage on the voltmeter will be satisfactory, provided the intended connections from this attenuator to other components are complete and the potentiometer is therefore correctly loaded. For precision in setting the coefficient value, a nulling arrangement must be used in which the voltage appearing at the output of the attenuator is nulled against the output voltage from an unloaded, carefully aligned, precision ten-turn potentiometer.

The nulling arrangement for an attenuator type 42.183 is shown in *Figure 2.* A patch cord must be connected from the arm of the attenuator to the VM-Jack on the control panel. A second cord must be used to connect +10V reference voltage to the "high end" of the attenuator. With the meter mode selector switch in the NULL position and the reference switch associated with the NULL POT switched to the +10 position, the attenuator can be adjusted until a null indication is obtained on the meter.

For the setting of attenuators Type 42.187, 42.188 and 2.128 (push button feature), a similar nulling arrangement is used with the exception that the operation of the push button will automatically accomplish the equivalent of the two cable connections mentioned above. The attenuator will be supplied with +10 volts, and its arm will automatically be connected to a POT BUS within the computer. Setting the meter mode selector switch to the POT BUS position will connect the nulling arrangement to the POT BUS. With the NULL POT reference switch in the +10 position, a null will be obtained by adjusting the attenuator.

When setting a potentiometer it is important that the load be the same during setting as during use in the problem. In practice the load on an attenuator usually consists of a resistor connected between the potentiometer arm and an amplifier summing junction which rides at ground potential. In some cases it is possible during the setting procedure to overload an amplifier so that its summing junction is no longer at ground. Should this happen, the overload alarm would sound and an incorrect setting would be made. To prevent this, the operator should ground the appropriate amplifier summing junction with a patch cord during the pot setting operation. The cord should be removed before proceeding with the problem.

In developing the electrical model of the primary physical system, a computer diagram is used and therefore a set of symbols is required to represent the different units. The symbol for an attenuator is shown in *Figure 3.* Note that the number of the attenuator is placed inside the circle, and the setting or adjustable parameter is indicated outside the circle.
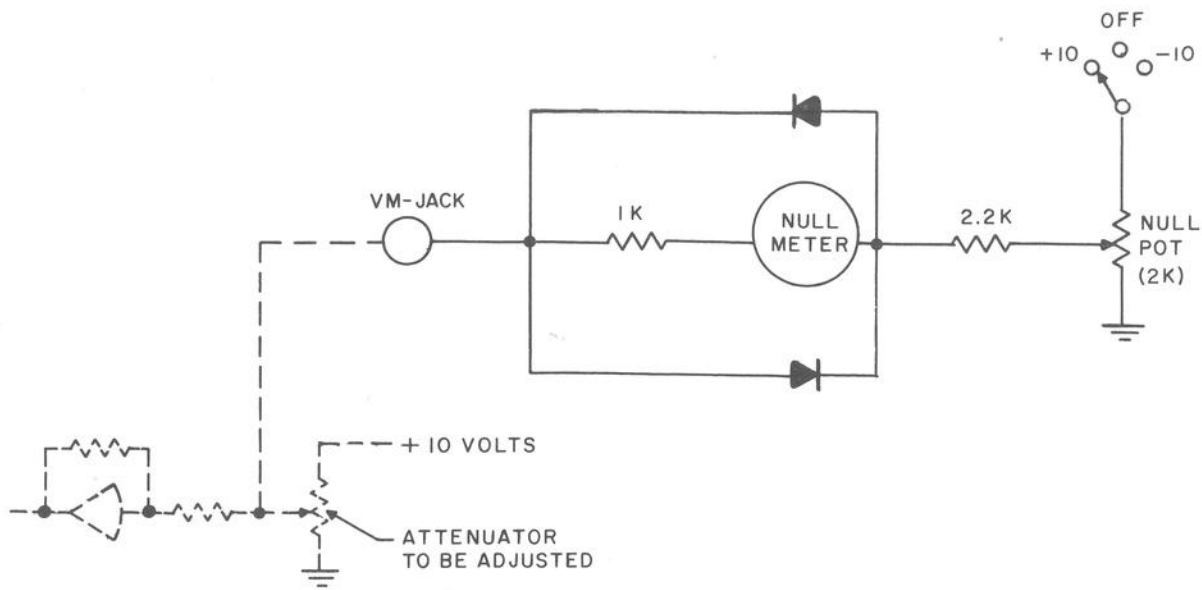
Figure 2. Nulling Arrangement

a) K = adjustable parameter (setting of attenuator
     may be changed for different problem runs).

b)   Indication of the actual numerical setting of
      the attenuator shall be used whenever the
      setting remains unchanged for a series of
      problem runs.

Figure 3. Attenuator Symbol

OPERATIONAL AMPLIFIERS TYPE 6.112-1.

The most versatile computing unit within the computer is the operational amplifier. It can be used in any
one of a number of ways to effect different mathematical operations. Sometimes, as will be seen, the opera-
tion requires the connection of other components to the amplifier. To appreciate the use of an operational

amplifier consider the circuit shown in *Figure 4*, where the symbol containing −A represents a high-gain
d-c amplifier. As is customary in all analog computing circuits, the voltages e_i, e_g, and e_o are measured
with respect to a common ground potential. Applying Ohm's law to the input of the amplifier, one has
summing currents:

$$\frac{e_i - e_g}{R_i} + \frac{e_o - e_g}{R_f} = i_g \tag{1}$$

Also, by definition of A:

$$e_o = - A e_g \tag{2}$$

Now i_g, the amplifier's input current, is extremely small compared with the currents flowing through
R_i and R_f given by the quotients involved in equation (1). Thus, one may approximate it as zero, and,
combining the two equations, one has:

$$e_o = - \frac{R_f}{R_i} \cdot \frac{e_i}{1 + \frac{1}{A}\left(1 + \frac{R_f}{R_i}\right)} \tag{3}$$

But A, though dependent on frequency, has a value in the range of $10^4 - 10^7$ for all expected computer
operations and, therefore, provided the ratio $\frac{R_f}{R_i}$ is comparatively small (less than 100, say), one can write:

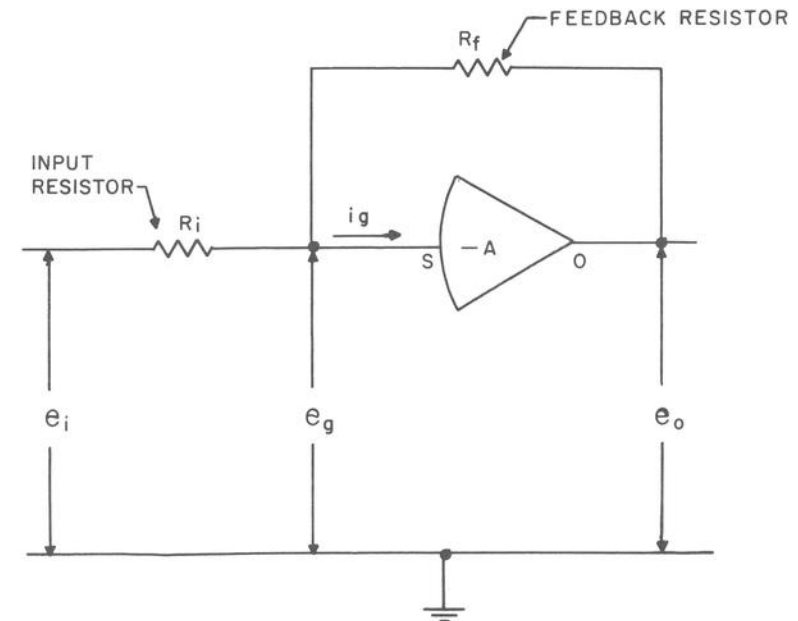$$e_o = - \frac{R_f}{R_i} e_i \tag{4}$$

Figure 4. Operational Amplifier

Thus one obtains the most important result that, provided the input current is low and the amplifier gain is large and negative, then the output-input relation is solely dependent on the ratio of the feedback to the input resistors. By suitable choice of resistors or other impedances such as capacitors, many worthwhile relations can be obtained. A few of the more commonly required ones will be detailed in the following paragraphs.

1. Sign-changing or Inversion

If both resistors are of equal value, 10K ohms say, the output voltage will be equal but opposite in sign to the input voltage.

$$e_0 = - e_i \tag{5}$$

Blue plugs containing 10K ohm resistors are available for connecting between the output and input terminations of a high-gain d-c amplifier, and single pin plugs containing 10K ohm resistors are available for use as input resistors. Thus the connection of a d-c high-gain amplifier for use as an inverter is as shown in the accompanying *Figure 5*. Two 100K resistors may also be used to make an inverter. However, this is not generally recommended since better performance is obtained with the 10K-10K inverter.

2. Multiplication by -10.

If the feedback resistor has ten times the value of the input resistor, then the output voltage will be op-
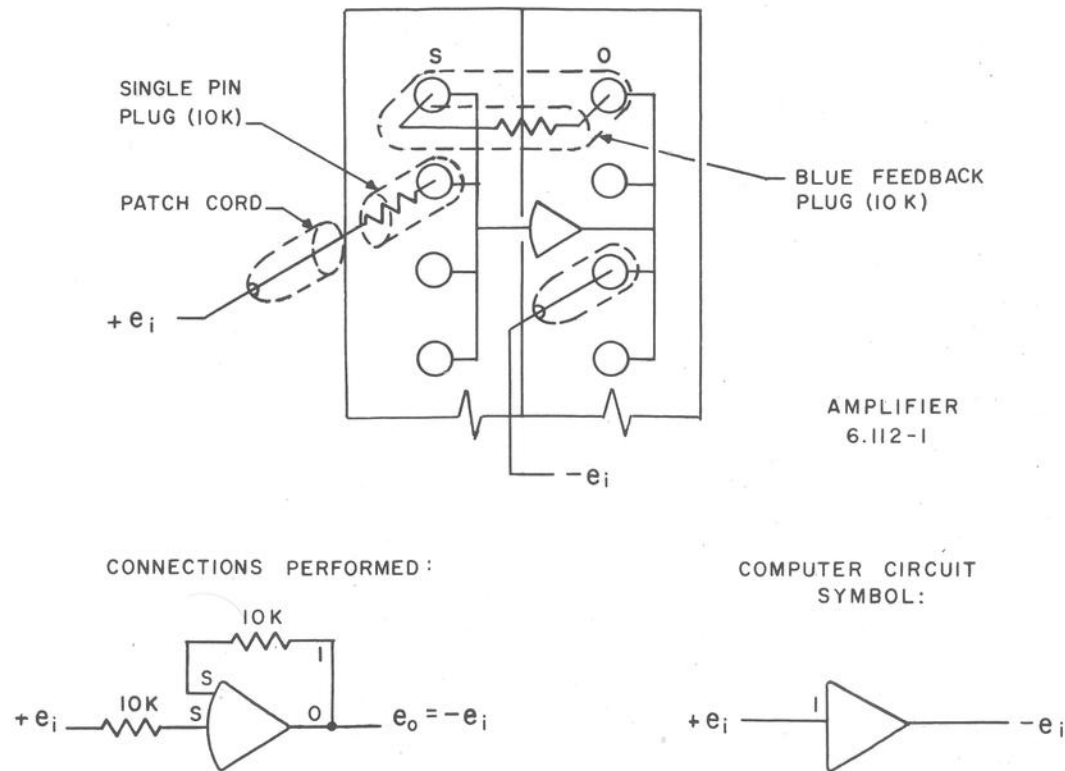
Figure 5. Operational Amplifier — Inversion

posite in sign and ten times the input voltage.

$$e_0 = - 10 \, e_i \tag{6}$$

Single pin 100K ohm plugs are available for use with the amplifiers and thus a connection as shown in the accompanying *Figure 6* effects a multiplication by −10.

3. Multiplication by a constant − a, 1 < a < 10.

By the combined use of an attenuator and an operational amplifier, a multiplication by a constant between −1 and −10 can be obtained. *Figure 7* shows the appropriate connections.

4. Division by a constant β.

Division by a constant β can either be treated as multiplication by $1/\beta$ or can be obtained with the circuit shown in *Figure 8*. This circuit satisfies the relation:

$$\beta \, \frac{e_0}{R_f} = - \frac{e_i}{R_i} \tag{7}$$

and thus:

$$e_0 = - \frac{1}{\beta} \, \frac{R_f}{R_i} \, e_i \tag{7a}$$
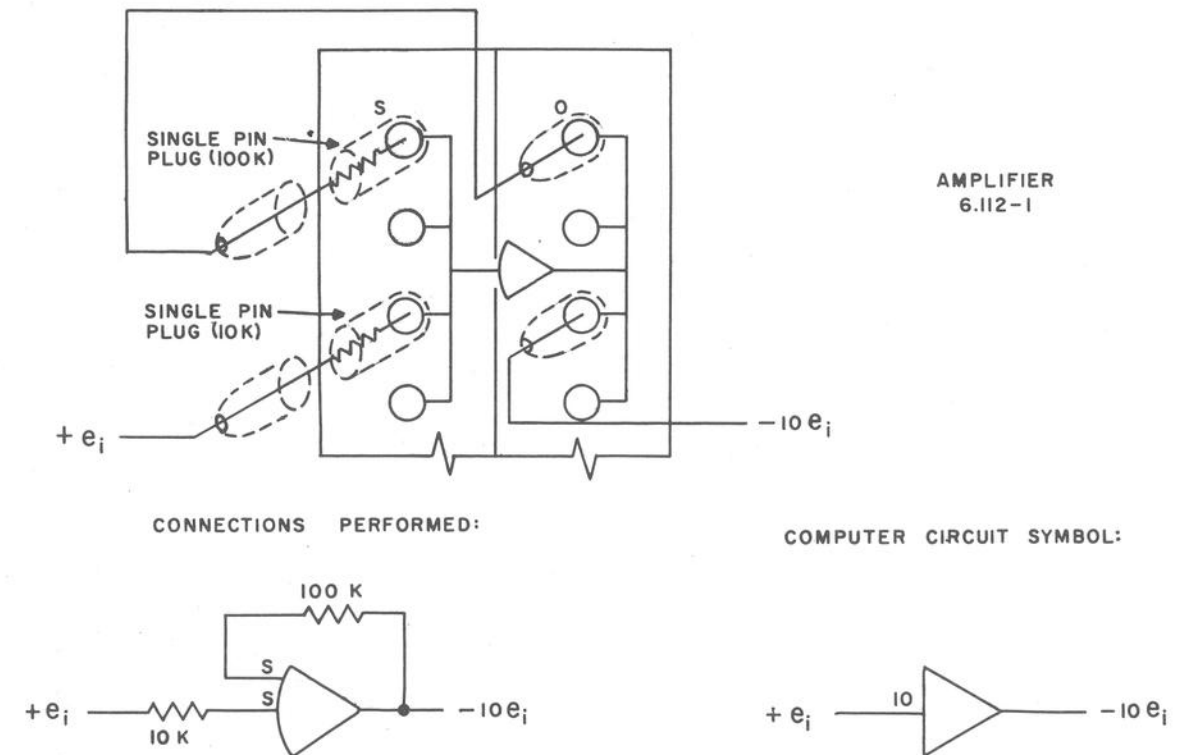
Figure 6. Operational Amplifier Multiplication by −10

ATTENUATOR

$+e_i$

AMPLIFIER
6.112-1

100 K

S   0

10 K

$-\alpha e_i$

CONNECTIONS PERFORMED

100 K

S

$+e_i$

SETTING
$\frac{\alpha}{10}$

10 K

S

$-\alpha e_i$

COMPUTER CIRCUIT

$+e_i$ ──── 10 ──── $-\alpha e_i$

$\frac{\alpha}{10}$
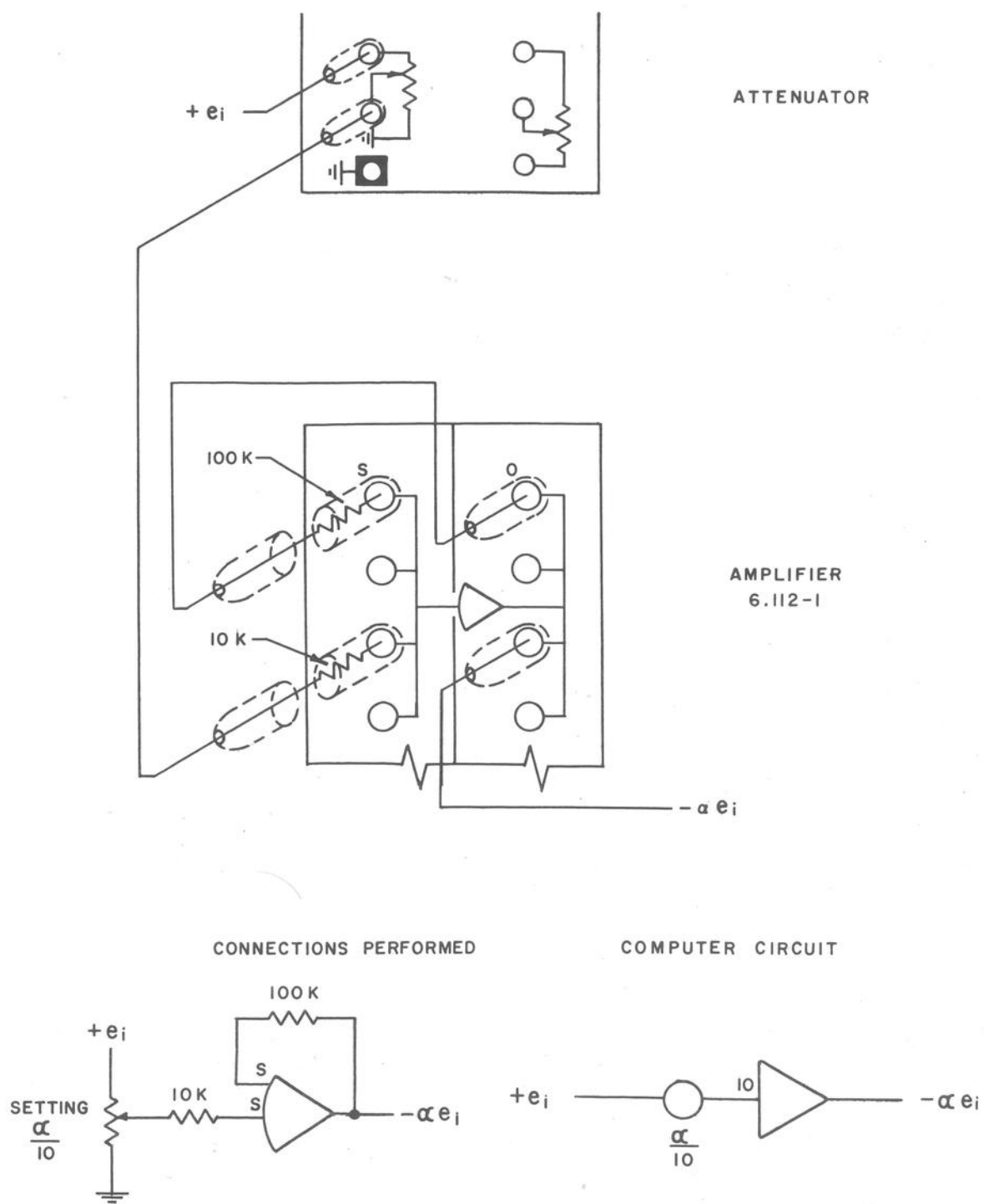
Figure 7. *Multiplication by a constant — α*

ATTENUATOR
42.183

AMPLIFIER
6.112-1

10 K

S   0

10 K

$+e_i$ $-\dfrac{e_i}{\beta}$

CONNECTIONS PERFORMED

$e_o \beta$

$R_f = 10K$ S   SETTING: β

$+e_i$ S   0

$R_I = 10K$ $-\dfrac{e_i}{\beta} = e_o$

COMPUTER CIRCUIT

β

1

1

$+e_i$ $-\dfrac{e_i}{\beta}$

Figure 8. *Division by a Constant $\div \beta$*

5. Summation of a Number of Voltages.

The inclusion of more than one input resistor to a high-gain d-c amplifier circuit, each resistor having a voltage applied to it, changes the original development of the input-output relationship to:

$$\frac{e_0}{R_f} = - \left[\frac{e_1}{R_1} + \frac{e_2}{R_2} + \frac{e_3}{R_3} + \ldots \right] \tag{8}$$

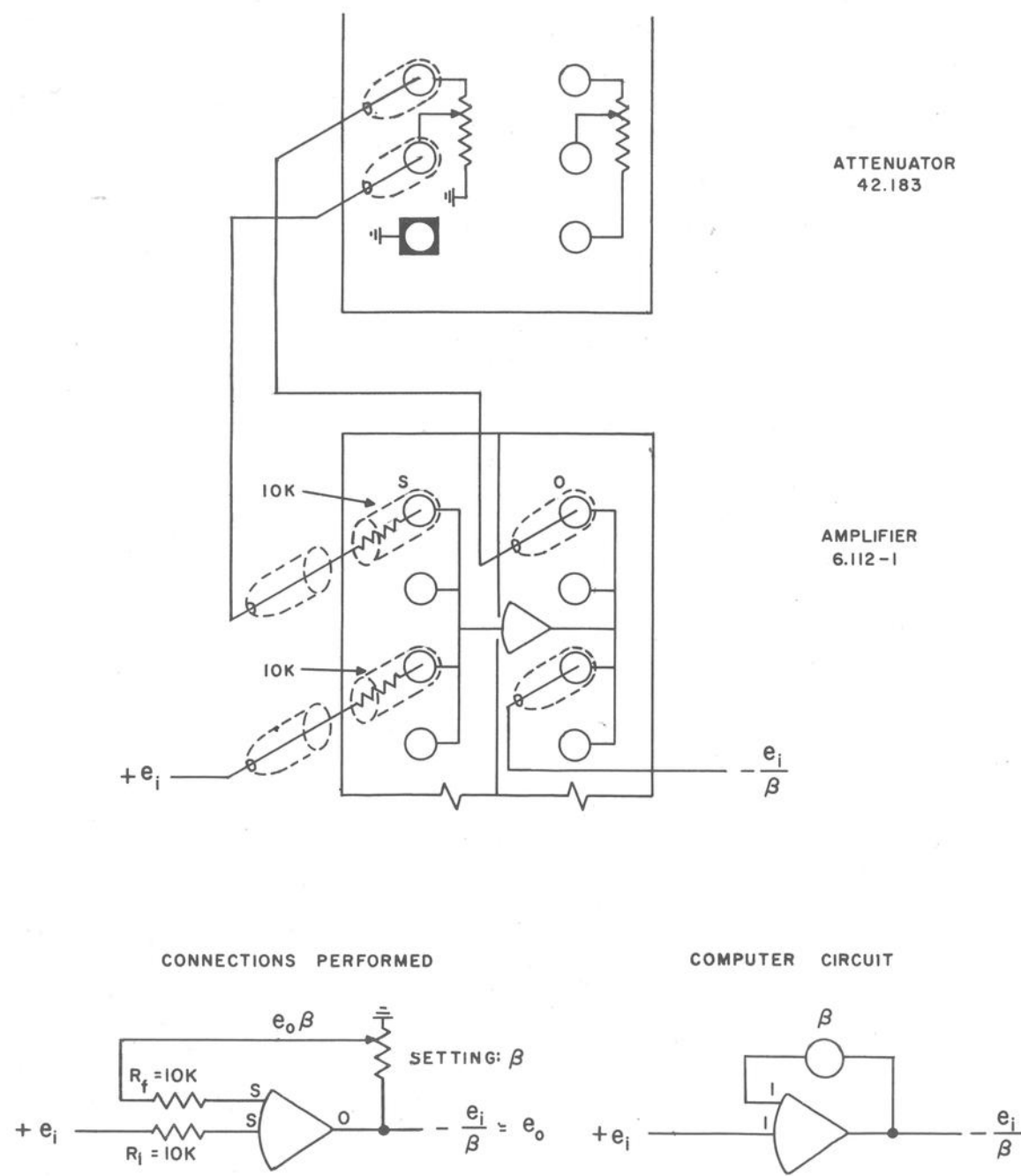That is to say the current flowing through the feedback resistor must be the algebraic sum of currents flowing through the input resistors since the amplifier input voltage and current are zero. Thus:

$$e_0 = - \left[\frac{R_f}{R_1} e_1 + \frac{R_f}{R_2} e_2 + \frac{R_f}{R_3} e_3 + \ldots \right] \tag{9}$$

By using equal values for all resistors, one obtains a simple algebraic summation with the usual inversion associated with every computing amplifier. If the resistors have different values, then each input voltage is multiplied by a factor given by the ratio of the feedback resistor to the input resistor before the sum is taken. If both positive and negative voltages are applied to an amplifier, then due recognition to sign is paid in the algebraic summation process. *See Figure 9.*
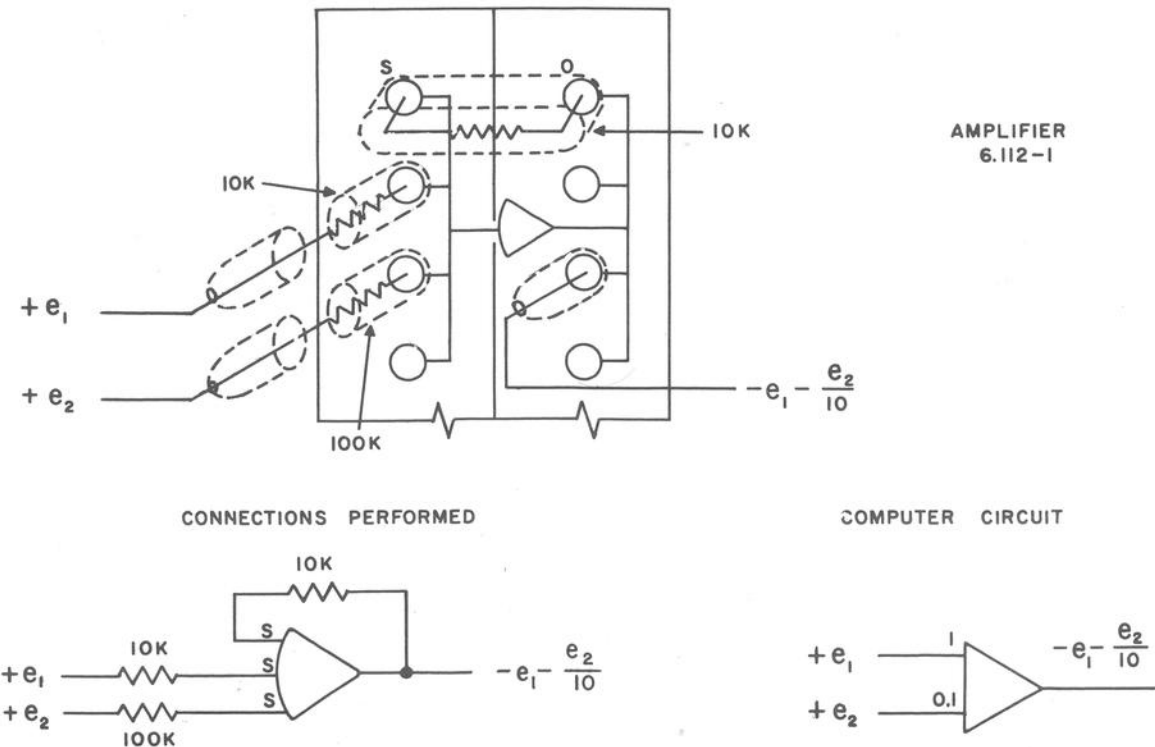

Figure 9. *Operational Amplifier — Summing*

6. Integration with respect to time.

If the algebraic sum of the input currents is forced to pass through a feedback capacitor rather than a resistor, the current equation is:

$$C \frac{de_0}{dt} = - \left[\frac{e_1}{R_1} + \frac{e_2}{R_2} + \frac{e_3}{R_3} + \ldots \right] \tag{10}$$

This relation depends simply on the fact that when current passes through a capacitor the voltage across the capacitor changes with a rate proportional to the current. Integrating equation (10) and assuming an initial voltage across the capacitor of V_0 gives:

$$e_0 = - \frac{1}{C} \int_0^t \frac{e_1}{R_1} + \frac{e_2}{R_2} + \frac{e_3}{R_3} + \ldots] \, dt \ + \ V_0 \tag{11}$$

If C has a value of 10 microfarads and all input resistors have values of 100K ohms, the expression for e_0 reduces to:

$$e_0 = - \int_0^t [e_1 + e_2 + e_3 + \ldots] \, dt \ + \ V_0 \tag{12}$$

If an input resistor has a value of 10K ohms, then the corresponding input voltage is multiplied by 10 before summation and integration.

It is necessary to be able to control the operation of integration and also to be able to apply an initial charge to the capacitor to set the initial value (V_0) of the output voltage e_0. Thus, to form an effective integrator more than a simple capacitive feedback is required. To effect integration on the TR-10, one connects an integrator network across a high gain d-c amplifier and the necessary input resistors are applied to that integrator network. *Do not apply input resistors to the standard input terminations of the d-c amplifier* for this would by-pass the most important "hold relay" in the integrator network (see later). The output voltage may be taken from either the network or the amplifier terminations. *(See Figure 10).*

With these six examples there has been demonstrated the versatility of the operational amplifier. To summarize, one might list the following operations as those performed by the unit when suitably connected with resistors and/or capacitors.

    a) Inversion

    b) Multiplication by a constant greater than unity

    c) Algebraic summation

    d) Integration with respect to time of an algebraic summation

The amplifier has uses other than those of directly effecting mathematical operations. By suitable analysis combinations of input and feedback elements can be determined for developing with one amplifier many desired transfer relations. Frequently the amplifier is used without a direct feedback connection through a passive element but with a feedback path through other computing components. With this arrangement the amplifier brings about a null of the input currents by changing that one dependent on the feedback connection. A simple example of this use appears later in the manual when the quotient of two variables is discussed.

COMPUTER SYMBOLS

Consistent with the standard symbols used in drawing computer circuits for the larger vacuum-tube com-
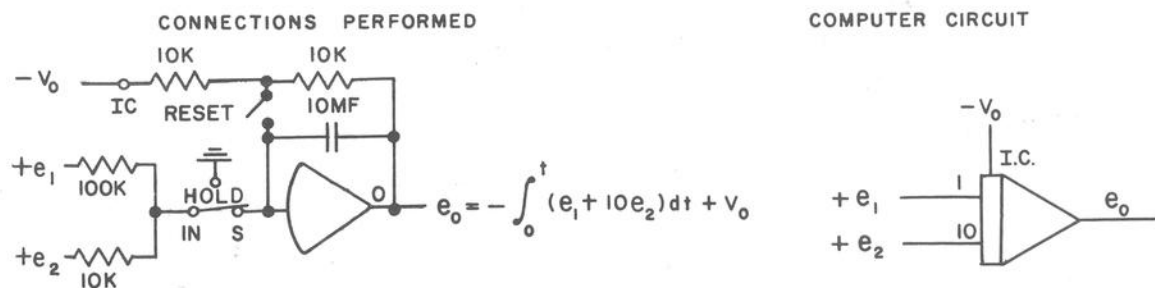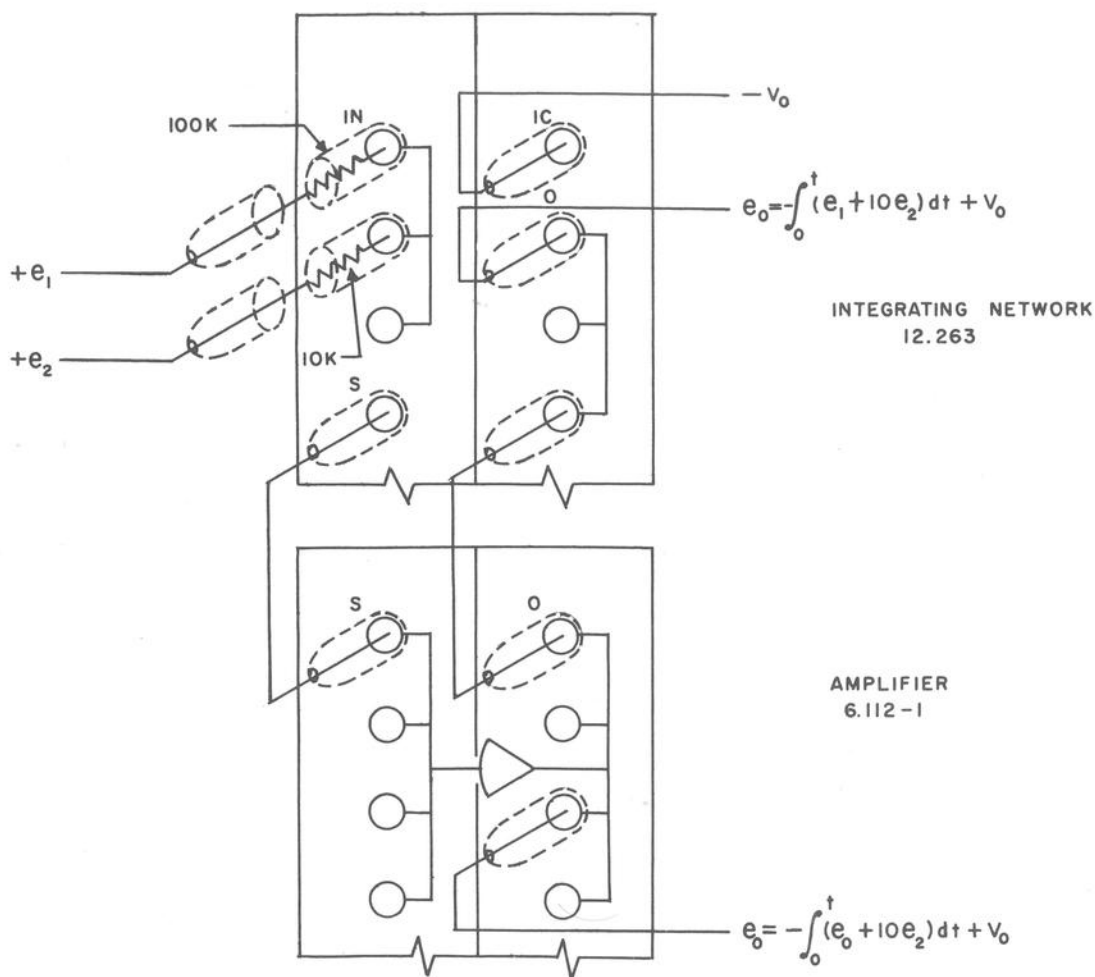
puters, the symbols used to represent summing, integrating and high-gain amplifiers are given along with those of other devices in Appendix I on page 67.

CIRCUIT INFORMATION REQUIRED FOR THE APPROPRIATE OPERATION OF THE HIGH-GAIN D-C AMPLIFIER

For accurate computation the amplifier must remain balanced. It must produce a zero output voltage when the combined effects of the input voltages or the absence of all input voltages demands it. With this requirement in mind the amplifiers contain chopper stabilization circuits to minimize the effects of component drift. Under normal circumstances the amplifiers will remain balanced for periods of hours or days. However, at intervals it is necessary to check this condition, and if an amplifier is found to be unbalanced then an adjustment is recommended.

Below the control panel, hidden by a snap-fit cover plate, are the balance potentiometers for the amplifiers. With the Meter Mode Selector Switch in the position "BAL", each amplifier can be selected on the Amplifier Selector Switch and its balance tested and trimmed. When making this adjustment, it is necessary for all amplifiers to have a feedback connection of some form.

In normal use the deflection of the balance meter should be adjusted to within two or three divisions of zero. When first turned on, the deflections may be slightly higher but will return to their normal levels after 30 or 60 minutes warm-up. For unusual problems which might be sensitive to amplifier unbalance or integrator drift, amplifiers may be balanced at more frequent intervals so as to keep the meter deflection below one division.

The operation of the computer is intended to accomodate signal voltages having values up to a maximum of the reference voltage $\pm 10V$. Under some conditions the amplifiers can produce output voltages as high as $\pm 14V$, but for accurate, reliable computation they should not be required to give more than $\pm 10V$. An overload alarm feature is included in the computer which indicates the presence of an overloaded amplifier, usually due to a patched circuit requiring too large an output voltage. The overloaded amplifier may be located by placing the meter select switch in the BAL (balance) position and rotating the AMP (Amplifier Selector) switch until a large meter deflection is noted. The overload alarm does no more than indicate that something is wrong with the patched circuitry, the problem operation or the components. It is well to note that occasionally an overloaded amplifier will cause no appreciable error in a problem solution and then there is no need to take any corrective action. However, the operator should ascertain that this is the case before proceeding with the problem.

MULTIPLIER, TYPE 7.045.

Multiplication of two variable voltages is a non-linear operation which is necessary on a general purpose computer. A "quarter-square" technique is used to effect this operation, use being made of the identity:

$$X Y = \frac{1}{4} \left[(X + Y)^2 - (X - Y)^2 \right] \tag{13}$$

The squaring operations are performed by fixed DFG's of the type described later, and interconnected circuitry within the unit requires only that the connections shown in *figure 11* be made to the patch panel terminations for this unit. All four inputs $(+X, -X, +Y, -Y)$ must *always* be patched in even though one or both of the inputs may not change sign during a problem run. Note that the output voltage from the required associate amplifier is $-XY/10$. A change in the output to $+XY/10$ is accomplished easily by interchanging the $+X$ and $-Y$ inputs, or $+Y$ and $-Y$ inputs.

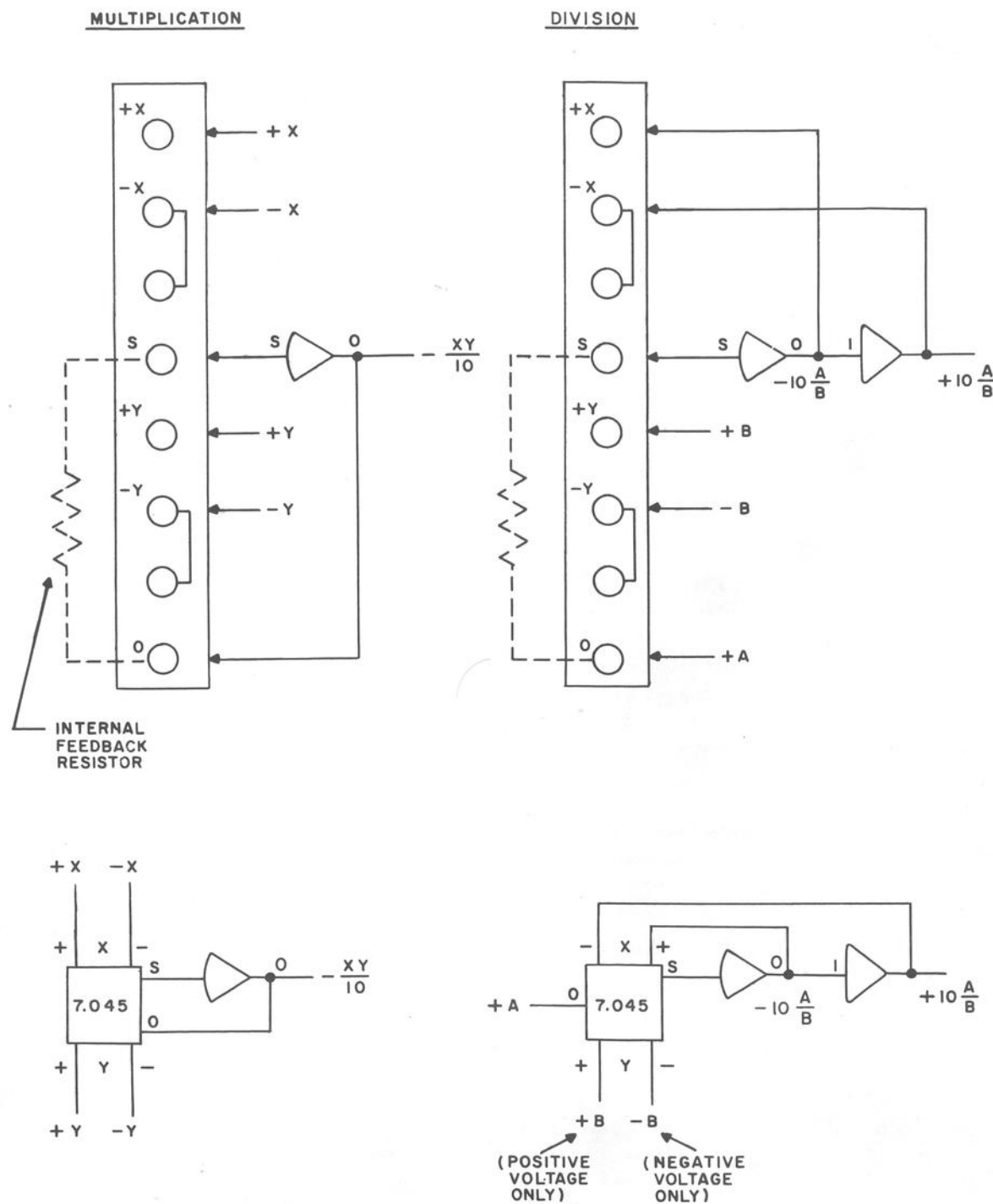An appropriate symbol for the multiplier is given in Appendix I, page 67.

Figure 10. Operational Amplifier — Integration

MULTIPLICATION   DIVISION

INTERNAL
FEEDBACK
RESISTOR

(POSITIVE   (NEGATIVE
VOLTAGE   VOLTAGE
ONLY)   ONLY)

Figure 11. Multiplier 7.045 Connections

16

DIVISION

When it is necessary to divide one variable voltage A by a second variable voltage B, one uses a multiplier in the feedback circuit of a high-gain amplifier. Consider the computer-circuitry shown in *Figure 11.* Assume the output voltage of the high-gain amplifier to be C. Then at the grid of the amplifier the null relationship

$$\frac{CB}{10} + A = 0 \qquad (14)$$

is satisfied by automatic changes in the value of C. Hence,

$$C = - \frac{10A}{B} \qquad (14a)$$

A most important point to note about this circuitry is that although the voltage A can have both positive and negative values, the B voltage must always have positive values. Should B always be negative, then the use of $-B$ in its place, (i.e., the interchange of the connections to $+y$ and $-y$), will produce at the output of the high-gain amplifier,

$$C = + \frac{10A}{B} \qquad (14b)$$

One must place the following restrictions on the voltages in a quotient :

1. The absolute value of the divisor B must always be greater than or equal to the absolute value of the dividend A, otherwise an overload may occur in the output amplifier.

2. The divisor B must not change sign. It must not pass through zero, for this would imply an indeterminate or infinite quotient. Moreover, the circuitry requires that the high-gain amplifier be surrounded by negative feedback, and this can only be arranged for one or other sign of the B voltage. If B is a negative voltage, then $+B$ and $-B$ are interchanged and the output of the high-gain amplifier is then $+ \frac{10A}{B}$ rather than $- \frac{10A}{B}$.

X² DIODE FUNCTION GENERATOR, TYPE 16.101.

As another example of the non-linear function generating equipment available in the computer, we will consider briefly the X² DFG. Using resistors and solid state diodes, this unit generates two quadratic curves, one for positive input voltages, the other for negative input voltages. Each curve is made of seven straight-line segments approximating the function X². When used with an operational amplifier the section terminated on the upper part of the X² DFG module accepts a negative voltage X and produces a positive voltage $X^2/10$. Should X be a positive voltage, the output will be zero. The section terminated on the lower part of the X² DFG module accepts positive voltages X and produces from the associated operational amplifier a negative voltage $-X^2/10$. Should X in this case be a negative voltage, the output will be zero.

The patching interconnections required to obtain various types of mathematical functions with the X² DFG are shown in the accompanying *figures 12* and *13*. The symbol for use in a computer circuit diagram is shown in Appendix I, page 67.

LOG DIODE FUNCTION GENERATOR, TYPE 16.126

A second frequently occurring non-linear function which can be generated with a component available in the computer is the logarithm. Employing the same kind of circuitry as the X² DFG, the LOG DFG ac-
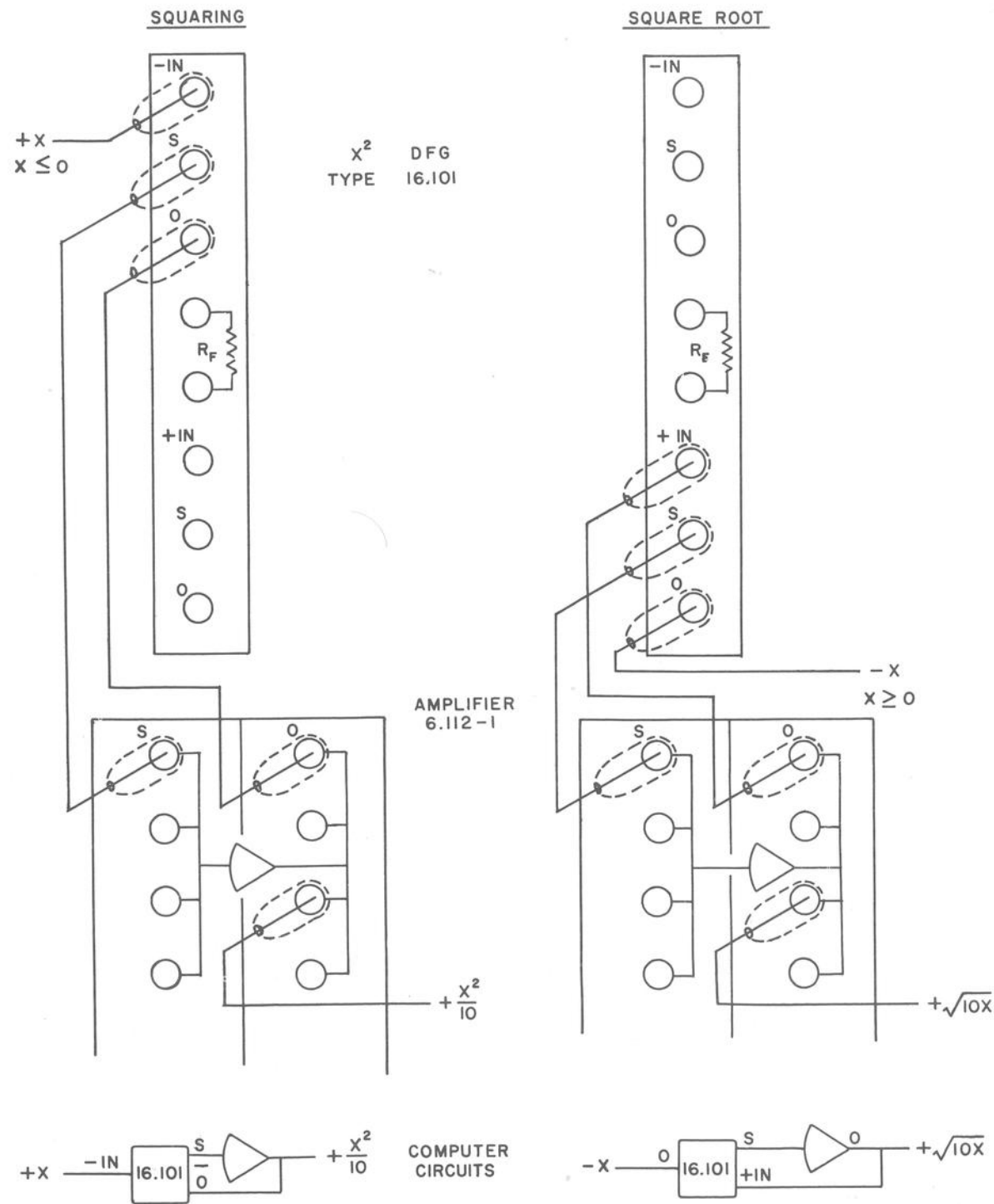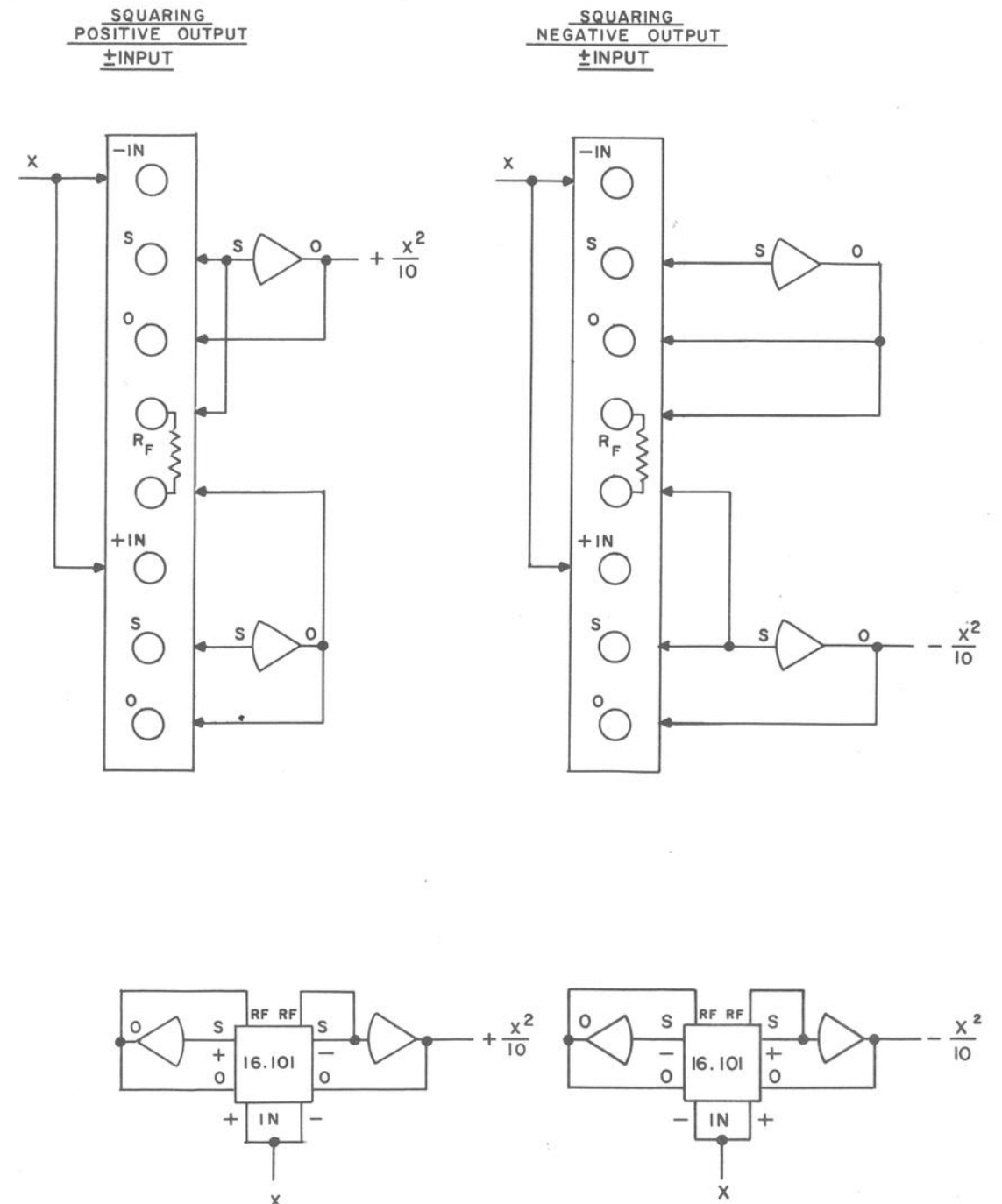
17

Figure 12. X^2 DFG Connections

Figure 13. X^2 DFG Connections

cepts a voltage X with magnitude between 0.1 and 10 and with the aid of an output amplifier produces a voltage proportional to $\text{Log}_{10}[10|X|]$. Each module contains two separate function generator circuits; one terminated on the upper half of the face of the module accepts negative voltages X and produces $Y = +5 \text{ Log}_{10}[10|X|]$, the other terminated on the lower half of the face of the module accepts positive voltages X and produces $Y = -5 \text{ Log}_{10}[10|X|]$. For either section the absolute value of X should normally be restricted to lie in the range $0.1 \leq |X| \leq 10$. Although input values between 0 and 0.1 may be used, accuracy limitations with low value inputs limit the usefulness of the answer.

The normally required connections for the unit are shown in *Figure 14.*

Just as the X^2 DFG can be used to produce its inverse, the square root, so the Log DFG can be used to produce the Antilog, see *Figure 15*.* By using three Log DFG's and one amplifier, multiplication of two variables, neither of which change sign, is achieved. Similarly, one can obtain the quotient of two variables neither of which change sign. The use of one more DFG section allows either the multiplication or the division of a quotient by a third variable.

VARIABLE DIODE FUNCTION GENERATOR 16.165

In many investigations the dependence of one variable quantity Y on another quantity X is known only in the form of an experimentally-determined curve. For example, the drag coefficient of an airfoil is determined from wind tunnel experiments to be dependent on the air speed, but the dependence is known only as a table of corresponding values or as a graph of drag coefficient versus Mach number. The specific heat or thermal conductivity of a plastic material might be non-linearly dependent on the temperature of the material and this dependence might well be known, not as a neat algebraic relationship, but as a graph of corresponding values. These coefficients and their dependence on the appropriate variable, be it velocity, temperature, or any other quantity, must be included in any computer study of the physical system. Their somewhat irregular dependence makes it desirable to have available in the computer a device which will accept a voltage representing, say, Mach number and give out a voltage representing the drag coefficient. For a different problem this same device will accept a voltage representing temperature and give a voltage representing thermal conductivity. The variable diode function generator is such a device, accepting a voltage X and when suitably adjusted, giving a voltage Y which is a required function of X; $Y = f(X)$.

Using the same techniques as the X^2 and Log DFG's, the Variable DFG permits any function of the input voltage X to be represented by a number of straight line segments. Solid state diodes are biased to change their state of conduction, each at a different value of X as X changes from −10 volts to +10 volts. The breakpoints (i.e. the points at which the diodes begin conducting) are at selected values of X and are not adjustable by the user. Where each diode conducts, it contributes a current to the output of the DFG unit which is proportional to the input voltage X. The constant of proportionality for each current is adjustable by a "SLOPE" potentiometer associated with each diode circuit. The sum of the currents contributed by the diodes is applied to the summing junction of an operational amplifier and thereby forced to pass through a resistor placed around the amplifier. Thus the output voltage of the amplifier changes with X along a sequence of straight line segments, the slope of any segment depending on the number of diodes that are conducting and the settings of the slope potentiometers associated with those diodes.

The variable DFG 16.165 has ten diodes with break points set at −9v, −7v, −5v, −3v, −1v, +1v, +3v, +5v, +7v, and +9v. Each has a slope potentiometer. The value of the output voltage $Y = f(X)$ at $X = 0$ can be set at any value between −10v and +10v by the PARALLAX potentiometer, and the slope of the curve at $X = 0$ can be set by the CENTRAL SLOPE potentiometer to have any value between plus and minus 2 volts per

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*Amplifier output signals are obtained by summing Amplifier input currents. The current output of the 16.126 Log DFG is $10^{-3} \log_{10}(10X)$ amperes.
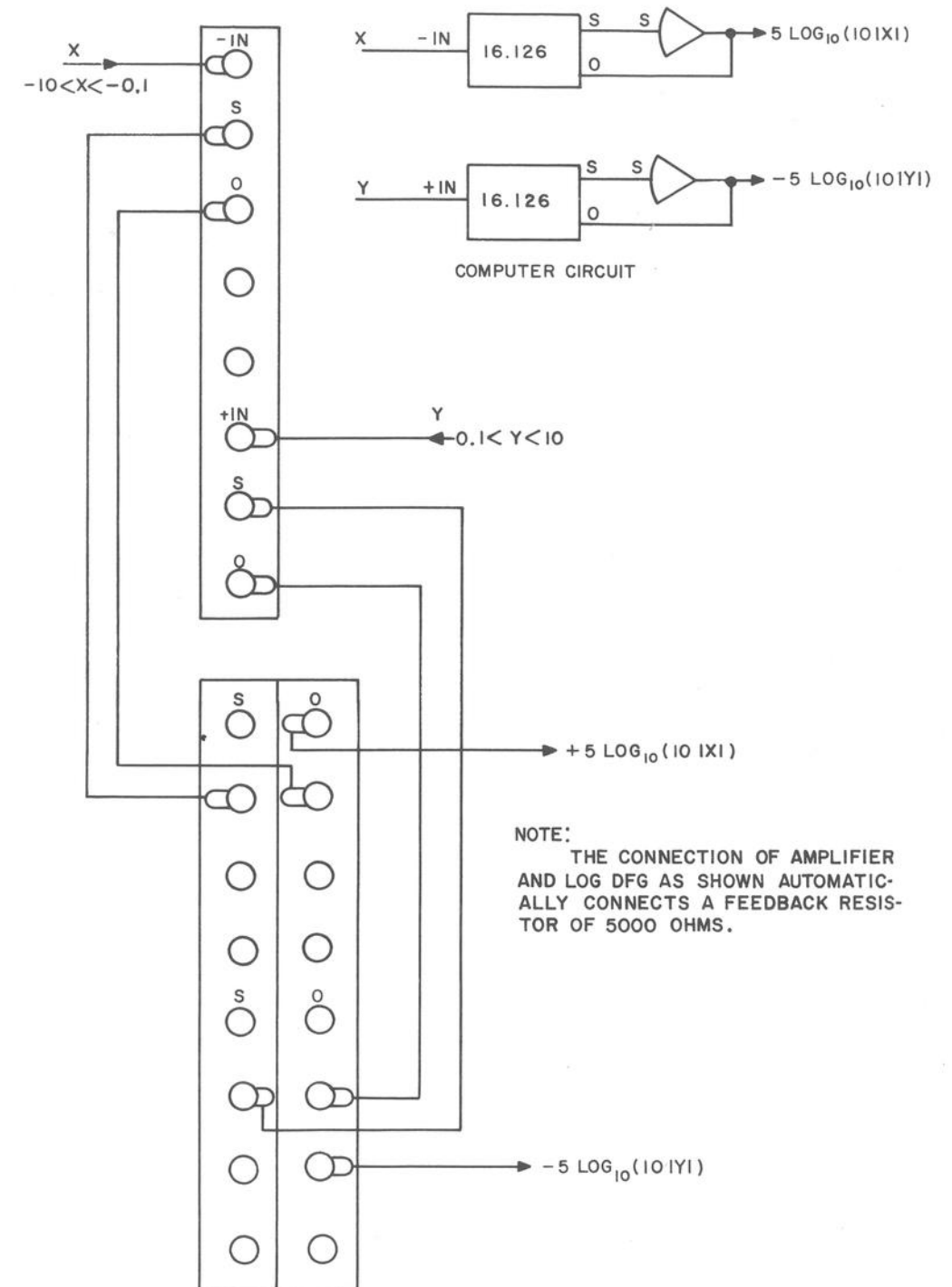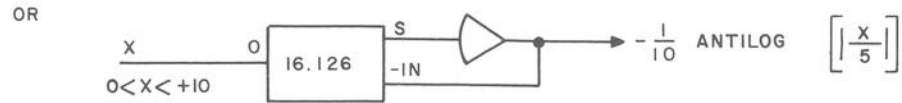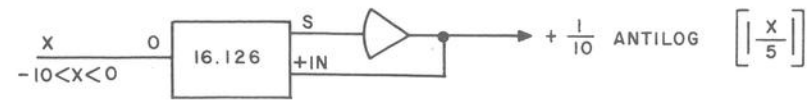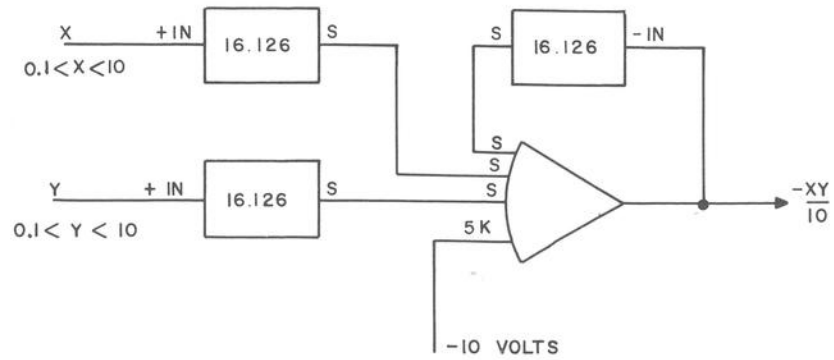
Figure 14. Log DFG Connections

NOTE:
THE CONNECTION OF AMPLIFIER AND LOG DFG AS SHOWN AUTOMATICALLY CONNECTS A FEEDBACK RESISTOR OF 5000 OHMS.

$$\frac{X}{-10<X<0} \quad \boxed{16.126} \quad S \quad \rightarrow + \frac{1}{10} \text{ ANTILOG} \left[\left|\frac{X}{5}\right|\right]$$

OR

$$\frac{X}{0<X<+10} \quad \boxed{16.126} \quad S \quad \rightarrow - \frac{1}{10} \text{ ANTILOG} \left[\left|\frac{X}{5}\right|\right]$$

(a)   ANTILOG CIRCUITS

$$\frac{X}{0.1<X<10} \quad +IN \quad \boxed{16.126} \quad S \qquad \boxed{16.126} \quad -IN$$

$$\frac{Y}{0.1<Y<10} \quad +IN \quad \boxed{16.126} \quad S \qquad 5K \qquad \rightarrow \frac{-XY}{10}$$

−10 VOLTS

OR

$$\frac{X}{-10<X<-0.1} \quad -IN \quad \boxed{16.126} \quad S \qquad \boxed{16.126} \quad +IN$$

$$\frac{Y}{-10<Y<-0.1} \quad -IN \quad \boxed{16.126} \quad S \qquad 5K \qquad \rightarrow \frac{+XY}{10}$$

+10 VOLTS
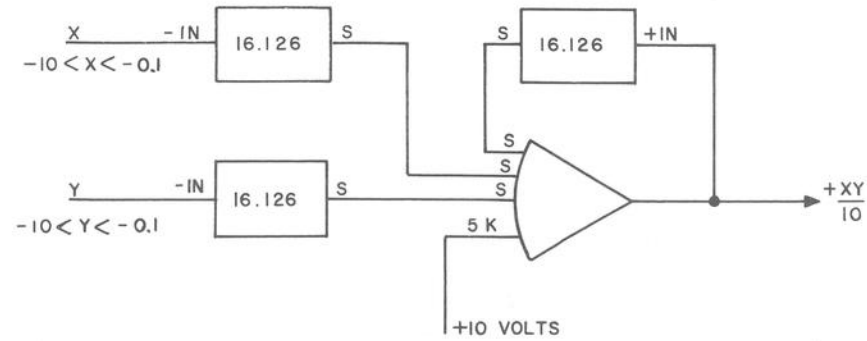
(b) MULTIPLICATION CIRCUITS

Figure 15. Log DFG Circuits

$$\frac{Y}{0.1<Y<10} \quad +IN \quad \boxed{16.126} \quad S \qquad S \quad \boxed{16.126} \quad -IN$$

$$\frac{X}{-10<X<-0.1} \quad -IN \quad \boxed{16.126} \quad S \qquad 5K \qquad \rightarrow + \frac{10\,Y}{X} *$$

$$|X| \geq Y$$

+10 VOLTS
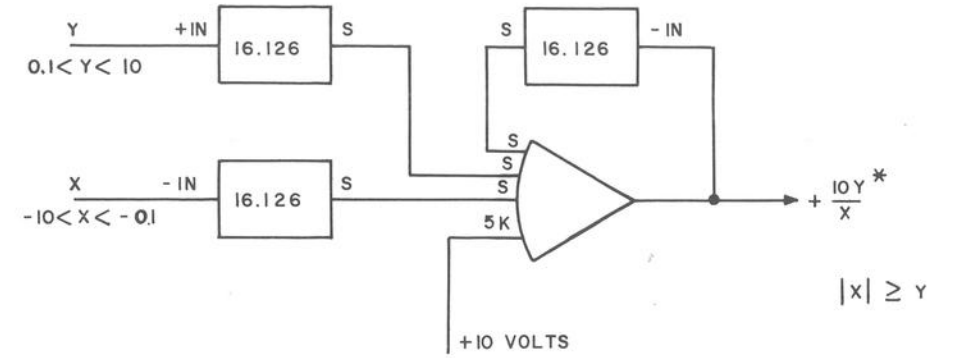
OR   * SINCE THE SIGN OF THE QUOTIENT $\frac{Y}{X}$ IS NEGATIVE, THE AMPLIFIER
VOLTAGE IS NEGATIVE.

$$\frac{Y}{0.1<Y<10} \quad +IN \quad \boxed{16.126} \quad S \qquad S \quad \boxed{16.126} \quad +IN$$

$$\frac{X}{10<X<-0.1} \quad -IN \quad \boxed{16.126} \quad S \qquad 5K \qquad \rightarrow \frac{-10\,X}{Y} *$$

$$Y \geq |X|$$

−10 VOLTS
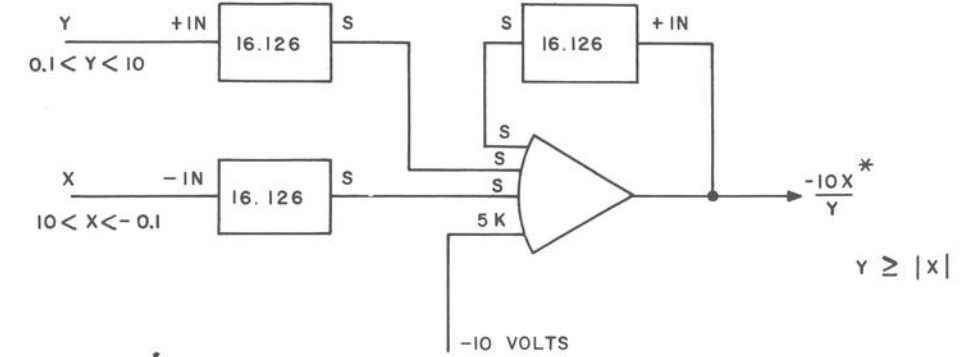
* SINCE THE SIGN OF THE QUOTIENT $\frac{X}{Y}$ IS NEGATIVE, THE AMPLIFIER
VOLTAGE IS POSITIVE.

(c) DIVISION CIRCUITS

$$\frac{Y}{0.1<Y<10} \quad +IN \quad \boxed{16.126} \quad S \qquad S \quad \boxed{16.126} \quad -IN$$

$$\frac{Z}{0.1<Z<10} \quad +IN \quad \boxed{16.126} \quad S \qquad \rightarrow + \frac{YZ}{X} *$$

$$\frac{X}{-10<X<-0.1} \quad -IN \quad \boxed{16.126} \quad S$$
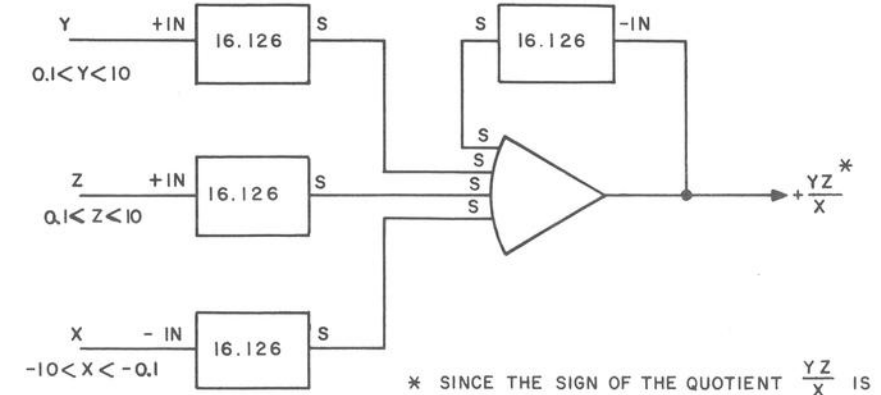
* SINCE THE SIGN OF THE QUOTIENT $\frac{YZ}{X}$ IS
NEGATIVE, THE AMPLIFIER VOLTAGE IS
NEGATIVE.

(d) COMPOUND MULTIPLICATION
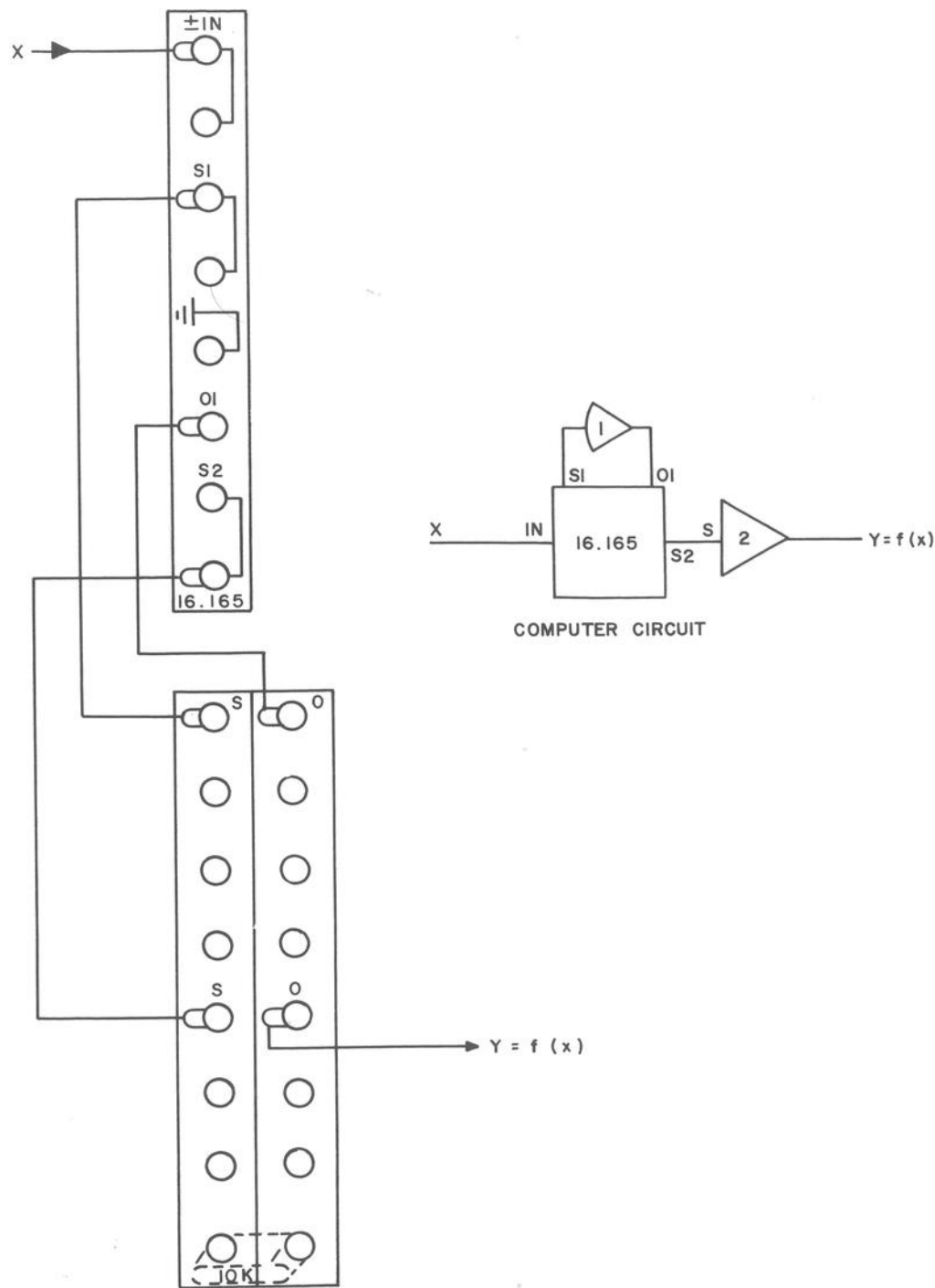
Figure 15. (continued)

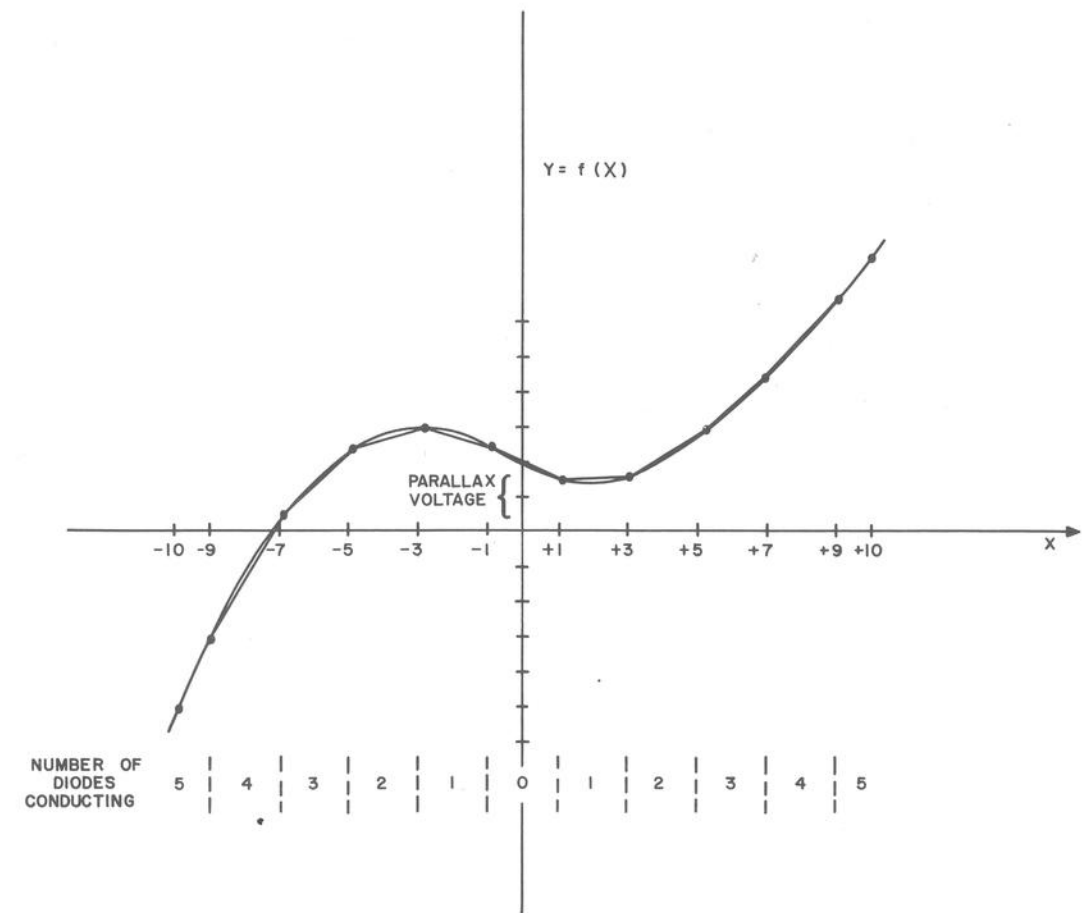Figure 16. *Variable DFG Standard Connections*

Figure 17. *Nonlinear Function Easily Generated by a Variable DFG*

volt. The slope potentiometer of each diode allows the slopes of adjacent straight line segments to differ by any value up to 1 volt per volt.

For the generation of any arbitrary function, two uncommitted operational amplifiers are required to be connected to the variable DFG as shown in *Figure 16*. This arrangement would permit the function shown in *Figure 17* to be generated quite easily.

The adjustment of the DFG to produce a desired function is facilitated by the use of an Attenuator unit type 42.243, which by connection to an inverting amplifier produces a voltage which can be accurately stepped from —10 volts to +10 volts in 1 volt increments. This permits the input voltage to be set conveniently to any of the breakpoint values, and then the slope potentiometers can be adjusted until the amplifier connected to the output of the VDFG reaches the desired voltage f(X). If no Attenuator unit is available the required breakpoint voltages have to be obtained from one of the common attenuators connected to an inverting amplifier.

The following steps should be followed in preparing a variable DFG to give a desired function *(see figures 18 and 19).*

1. Prepare a table of the appropriately scaled voltages desired at the output of the DFG when the input voltage has each of the following values: −10, −9, −7, −5, −3, −1, 0, +1, +3, +5, +7, +9 and +10 volts. Note that the values at −1, 0 and +1 must be colinear.

2. Plug the VDFG into a service shelf and connect the required amplifiers appropriately to the terminations of the unit. The VDFG potentiometers are now available for screw driver adjustment. Turn all potentiometers fully counter-clockwise.

3. Ground the VDFG input or set the input to zero and, by adjusting the PARALLAX potentiometer,
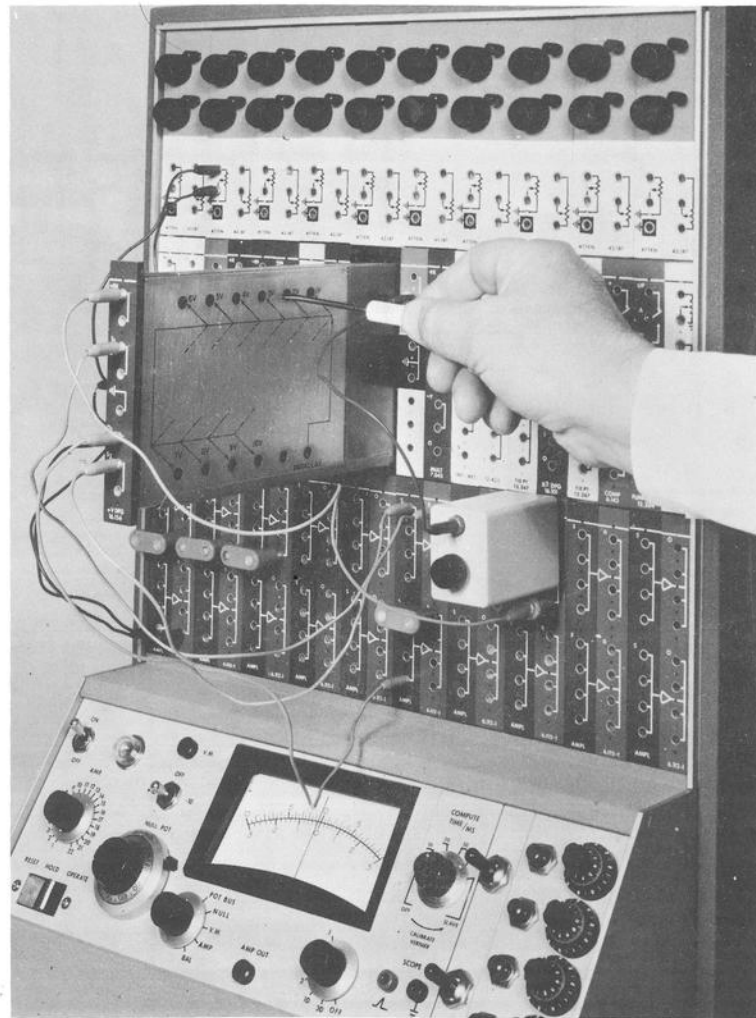

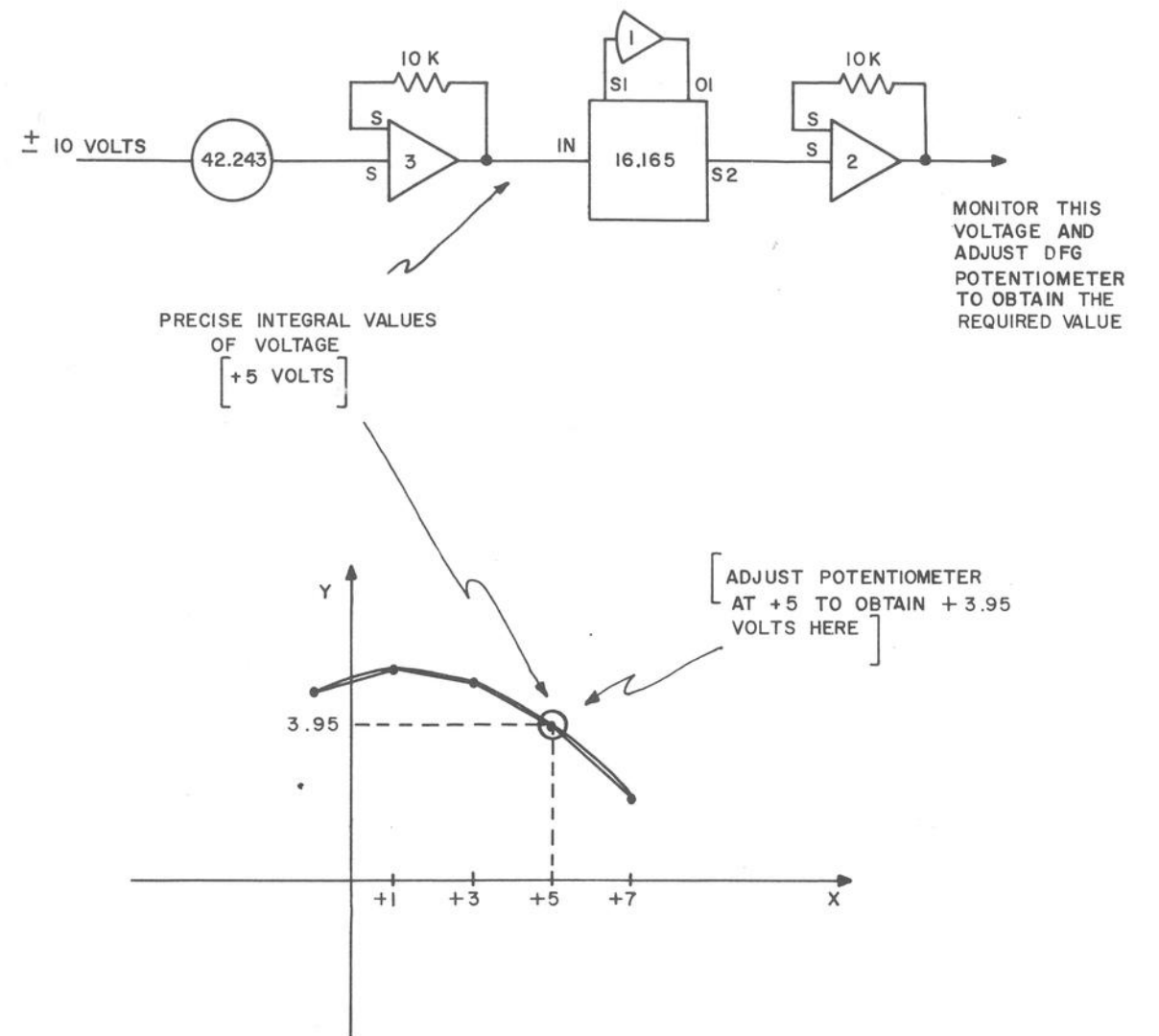*Figure 18. Patch Panel Arrangement for Adjusting a Variable DFG*

Figure 19. Circuit for adjusting a Variable DFG

obtain the appropriate output voltage f(0) from amplifier 2. Use the procedure detailed later on page 32 to check this voltage.

4. Apply −1 volt to the VDFG input and adjust the potentiometer labeled "−1" until the output of amplifier 2 reaches the value f(−1).

5. Apply −3 volts to the VDFG input and adjust the potentiometer labeled "−3" until the output of amplifier 2 reaches a value f(−3).

6. Continue to change the input voltages and by adjusting potentiometers obtain the appropriate values at the output of amplifier 2 as listed in the following table:

Input Voltage	Adjust SLOPE potentiometer labeled	To obtain amplifier 2 output voltage
−5 volts	−5 volts	f(−5)
−7 volts	−7 volts	f(−7)
−9 volts	−9 volts	f(−9)
−10 volts	−10 volts	f(−10)
+3 volts	+3 volts	f(+3)
+5 volts	+5 volts	f(+5)
+7 volts	+7 volts	f(+7)
+9 volts	+9 volts	f(+9)
+10 volts	+10 volts	f(+10)

7. Quickly check through all points to ensure that the complete function is well adjusted.

8. Remove the service shelf and insert the VDFG into its operating position.

It is well to note that for certain functions which are monotonic and for which the slope is always increasing with the absolute value of X { e.g. X^3, e^x, Tan X }, the VDFG requires only one amplifier. The appropriate connections for this situation are shown in *Figure 20*.

VARIABLE DIODE FUNCTION GENERATORS 16.154 AND 16.156

The variable DFG 16.154 accepts only negative input voltages ($−10 \le X \le 0$). The variable DFG 16.156 accepts only positive input voltages ($0 \le X \le +10$). Each has nine diodes with breakpoints set at integral values of voltage 1 through 9. Its operation is quite similar to that of the type 16.165 and the required connections and adjustment procedures correspond closely. In the adjustment procedure the PARALLAX and ZERO SLOPE settings are made first, before proceeding to the diode SLOPE settings which are made in order. When obtaining monotonic functions the "one amplifier" connection shown in *Figure 20* may also be used with VDFG's 16.154 and 16.156.

In some cases it may prove desirable to generate a function over the range $−10 \le X \le +10$ with more accuracy than can be obtained with VDFG 16.165. By connecting a 16.154 and 16.156 in parallel the necessary function can be generated with 19 segments rather than with the 11 segments available with 16.165.

MISCELLANEOUS DEVICES AND PATCH PANEL TERMINATIONS.

In addition to the standard computing components, a number of devices are found to be useful in programming a problem investigation on the computer. They can be used by the operator as he desires and are reviewed here to complete the list of items terminated at the patch panel.

1. Dual Function Switch, Type 2.127.

Switches mounted on the control panel are terminated at points in the middle row of the patch panel. Almost self-explanatory in their use, these function switches are single-pole, double-throw with an OFF (disconnected) center position.

2. Signal Voltage Comparator, Type 6.143.

An automatic switching device, the comparator operates two sets of contacts, permitting computed prob-
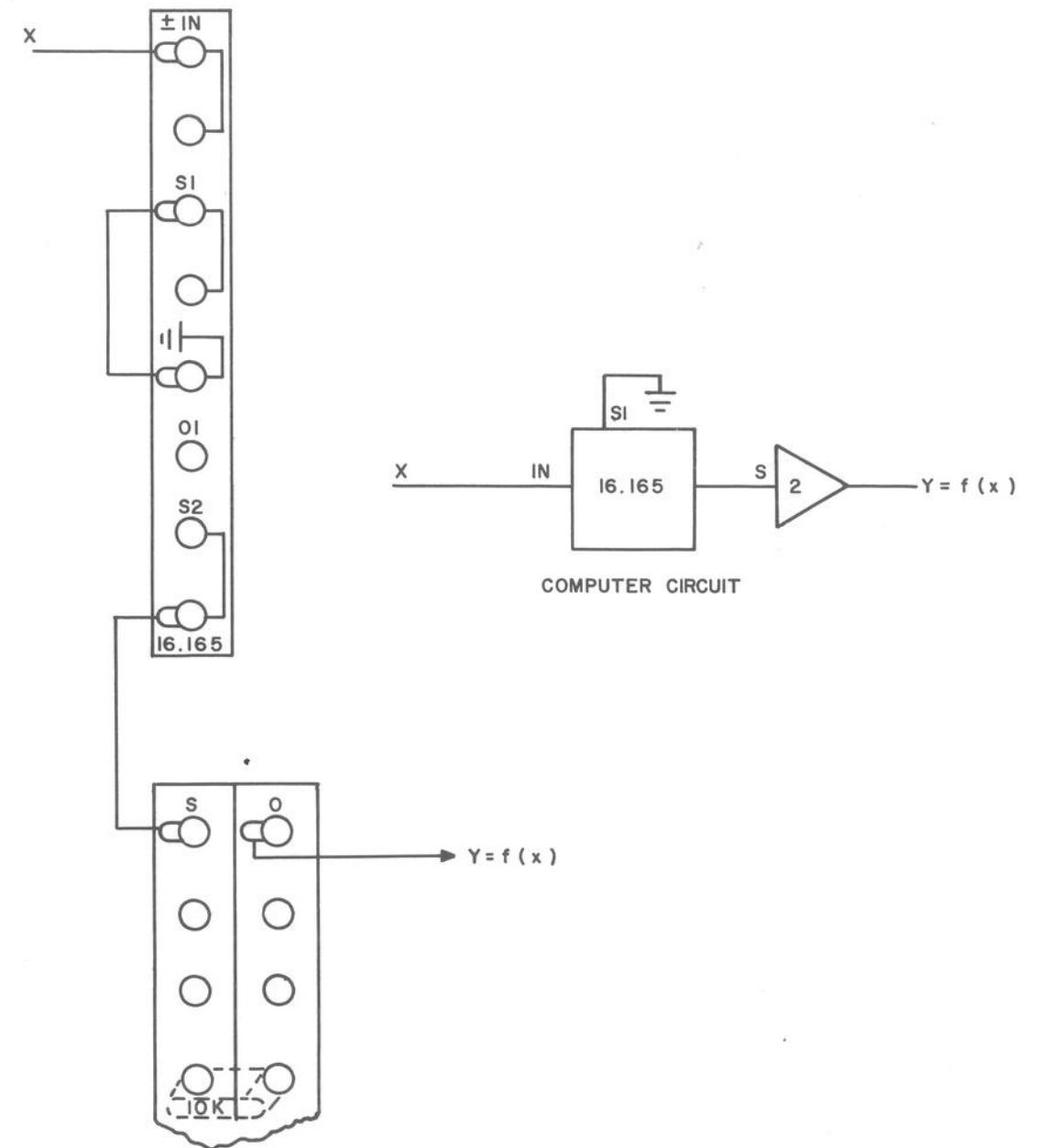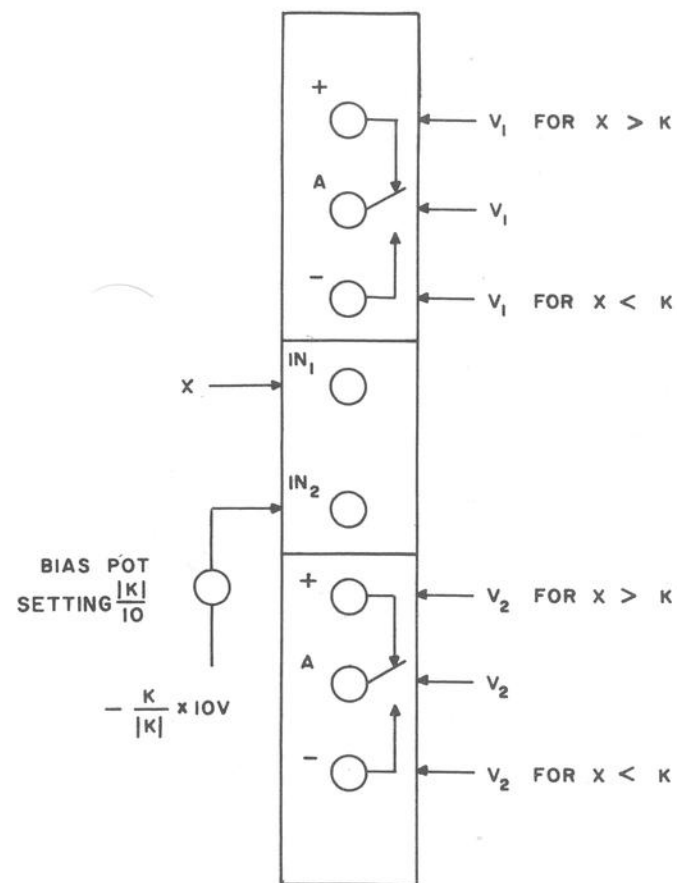
Figure 20. Variable DFG Connections for Monotonic Functions

lem voltages to determine connections or conditions applying in a patched circuit. As its name implies, the comparator accepts two input voltages, compares their sum to zero (approximately) and positions two switches up or down, depending on whether the sum is greater than or less than zero. The details of the unit's terminations are given in *Figure 21*.

An appropriate symbol to represent the comparator in circuit diagrams is given in Appendix I, page 67.

+

V_1 FOR X > K

A

V_1

−

V_1 FOR X < K

X → IN$_1$

IN$_2$

BIAS POT
SETTING $\frac{|K|}{10}$

+

V_2 FOR X > K

A

V_2

$-\frac{K}{|K|} \times 10V$

−

V_2 FOR X < K

Setup procedure.

1. Patch appropriate reference voltage through coefficient potentiometer into IN_1.

2. Adjust this potentiometer until its output reads desired switching voltage K.

3. Patch reference voltage through bias potentiometer into IN_2.

4. Adjust this potentiometer until relay switching occurs. Bias (IN_2) input is now set for proper switching level.

5. Remove coefficient potentiometer from IN_1 and patch variable X into IN_1.

6. The relay contacts will be in the positive position when $X > K$, they will be in the negative position when $X < K$.

Figure 21. Signal Voltage Comparator

---

3. Overload Alarm, Type 13.012.

This unit provides an audible warning signal of about 400 cps when an overload occurs in any of the operational amplifiers, i.e. when the summing junction error exceeds a safe level for any cause. When the alarm operates, the overloaded amplifier can be located quickly by setting the meter selector switch to the BAL (balance) position and rotating the AMP (Amplifier Selector) switch until a large meter deflection is observed.

4. Reference Voltage Supply.

The computer uses plus and minus 10 volts as sources for all computer signal voltages. Balanced about ground potential, these sources are available at the patch panel and are also connected internally to many points throughout the computer.

CONTROL PANEL 20.344.

In order to allow simple control and monitoring of the computing components, this panel contains the following components:

1. *Primary power switch.* Positions: ON, OFF.

2. *Neon light.* This light works in conjunction with the primary power switch and indicates when the computer is switched on.

3. *Mode Control Switch.* Positions: RESET, HOLD, OPERATE.

In RESET any circuit patched on the computer is functioning except that the outputs of all integrators are held at their required initial conditions. The programmed problem is thus set to those conditions corresponding to time zero.

By switching to OPERATE, the integrators are simultaneously set free and with the voltages applied to their input resistors causing changes in the output voltages, a time-varying behavior is produced. This will be the voltage solution of the programmed problem.

Switching to HOLD permits the solution to be held or "frozen" at any time that is convenient. After making whatever observations that are desired, the solution can be resumed by switching back to OPERATE or returned to its initial conditions by switching to RESET.

4. *Voltmeter.* Used as a voltmeter, null meter or balance indicator.

5. *Meter Range Switch.* This switch permits the selection of the following meter ranges when the meter is used as a voltmeter (i.e., in the AMP or V.M. positions of the Meter Mode Selector Switch):

± 30 volts
± 10 volts
± 3 volts
± 1 volt
± 0.3 volts
± 0.1 volts

6. *Meter Mode Selector Switch.* Positions: POT BUS, NULL, V.M., AMP, BAL.

POT BUS — Connects the meter as a Null meter to the computer Pot Bus for setting attenuators 42.187, 42.188 and 2.128 (push button feature).

NULL — Connects the meter as a Null meter to the *VM jack* connection on the control panel. This po-

sition is used for setting attenuators 42.183, which do not have the push button feature, and for measuring accurately other output voltages in the computer.

V.M. — Connects the meter as a voltmeter to the *VM jack*. Voltages patched to the VM jack may thus be read on the meter. Different meter ranges may be obtained with the meter range switch.

AMP — Connects the meter as a voltmeter to the AMPLIFIER SELECTOR SWITCH so that the output voltage of each amplifier can be read on the meter.

BAL — Connects the stabilizer outputs of the operational amplifiers, as selected by the AMPLIFIER SELECTOR SWITCH to the meter, which is now connected as a high sensitivity voltmeter. Thus, in this position, the stabilizer outputs of all amplifiers may be read on the meter and balanced to zero, through the use of the amplifier balance adjustments.

7. *NULL POT and Reference Switch.* Used to apply a precision voltage to the Null meter for setting attenuators or reading output voltages accurately. The reference switch controls the polarity of the voltage applied to the NULL POT (± 10 volts). The switch should be in the positive position for nulling positive voltages or setting attenuators and the negative position for nulling negative voltages. The NULL POT and reference switch operate in conjunction with the POT BUS and NULL positions of the meter selector switch.

8. *Amplifier Selector Switch (AMP).* The first 20 positions of the switch permit the selection of the outputs of the operational amplifiers. The voltages at these outputs may be read on the meter by placing the METER MODE SELECTOR switch in the AMP position. A second deck on the amplifier selector switch allows the selection of the amplifier stabilizer outputs which may be balanced to zero with the METER MODE SELECTOR switch in the BAL position.

Switch position 21 selects the positive reference voltage (+10V). Switch position 22 selects the negative reference voltage (−10V).

9. *VM Jack.* This terminal is connected to the meter when the METER MODE SELECTOR switch is in the NULL or VM position.

10. *AMP OUT Jack.* Amplifier outputs, as selected by the AMPLIFIER SELECTOR SWITCH, are available at the jack at all times for metering or recording purposes.

As pointed out in Paragraph 6 above, any steady voltage existing in the computer circuit may be read to three place accuracy by using the NULL POT in conjunction with the NULL position of the Meter Mode Selector Switch.

EXAMPLE: To read the output of an amplifier to three place accuracy :

  a. Connect AMP OUT Jack by means of a patch cord to VM Jack.

  b. Select amplifier output with the AMPLIFIER SELECTOR switch.

  c. Switch METER MODE SELECTOR switch to the NULL position.

  d. Observe deflection of meter. If needle deflects to the right (positive), switch null pot reference switch to +10. For a negative deflection switch null pot reference switch to −10.

  e. Adjust NULL POT until a null indication is obtained on the meter.

  f. To obtain the value of the amplifier's output voltage, multiply the NULL POT dial reading by the reference voltage used in step *d*.

REPETITIVE OPERATION

Repetitive Operation is an available feature which is most useful when solving certain kinds of problems. The problem solution time is reduced by a ratio of 100:1 and the computer automatically alternates between the RESET and OPERATE modes, causing the time behavior of interest to be produced many times each second. A solution which is programmed under normal operation to be obtained in 5 seconds, is achieved in 50 milliseconds, permitting it to be repeated approximately twenty times each second. The high-speed solution can be viewed on an oscilloscope and adjustments made to the potentiometer settings, their effect being seen immediately. Quickly a desired behavior can be sought, and then by returning the computer to normal operation it can be permanently recorded at slower speed on a plotting table. The time taken for one solution to be displayed is so short compared with the standard human recognition
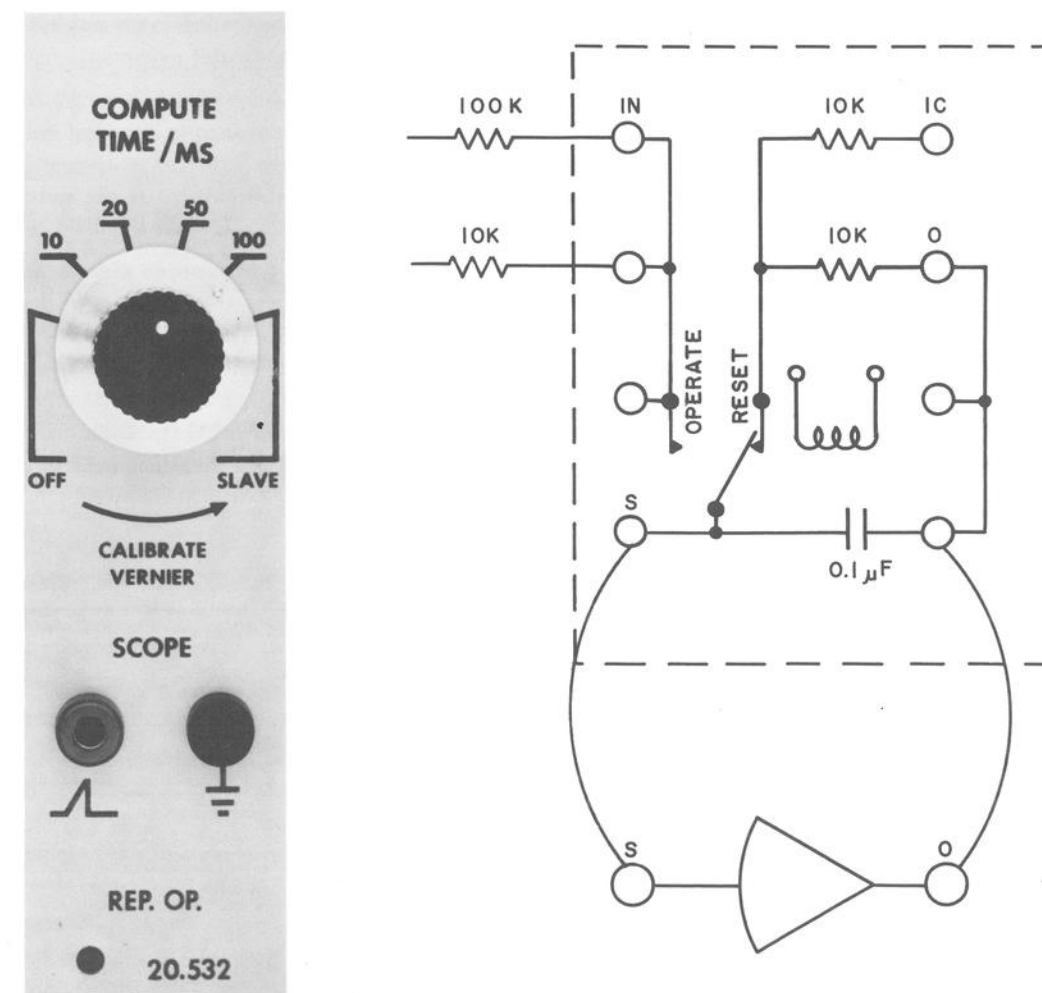
a) Control Unit Type 20.532   *b) Integrator Network Arrangement for Repetitive Operation*

Figure 22.

time that changes in behavior from solution to solution appear to be continuous; the behavior appears to sweep towards that desired.

To achieve repetitive operation three units are required.

    1. A control unit type 20.532, which fits into the computer's control panel. It contains a multi-position switch, which switches the computer over to repetitive operation and also determines the total time of solution — 10, 20, 50 and 100 millisecs. A vernier adjustment permits the time of solution to be further adjusted between these values; with the multi-position switch turned to 20, say, moving the vernier knob from its fully counter-clockwise position changes the time of solution gradually from 20 to more than 50 milliseconds. The control unit has a termination at which a sweep voltage for any standard oscillo-scope is available. The use of this voltage eliminates difficulties in synchronization that would other-wise need to be overcome. *(See Figure 22a.)*

    2. A timing unit type 36.082 which fits in the back of the console next to the reference power supply, and supplies the alternating, variable duration, relay control signals to the integrator networks.

    3. A set of integrator networks type 12.425 which in place of the integrator networks type 12.263, contain both 10 microfarad and 0.1 microfarad capacitors permitting the 100:1 change in speed of solution. These units have the same terminations as the type 12.263, illustrated earlier. With the multi-position switch of the control unit switched off the 10 microfarad capacitors are in the circuit. With the multi-position switch in any other position, the 0.1 microfarad capacitor is in the circuit. *(See Figure 22b.)*

In place of the standard "HOLD" and "RESET" relays shown in *Figure 10,* a high-speed electro-mechani-cal relay, controlled from the timing unit, switches the summing junction between the mid-point of the 10 K ohms initial condition resistors (RESET) and the input resistors (OPERATE). The integrators are RESET for 10 milliseconds between each solution. The time for which the integrators OPERATE is controllable between 10 milliseconds and approximately 250 milliseconds. During the setup of a problem when at-tenuators are being adjusted the Rep Op control switch should be turned to OFF so that internal relays will ground the summing junctions. If this is not done, loading errors will cause the attenuators to be set incorrectly.

II. BASIC PROGRAMMING

The earlier sections of this manual emphasized the use of the basic computing elements of the TR-10.

An integrated review of the steps necessary to program a problem for analog computation will now be dis-cussed.

The general procedure is:

    1. Obtain mathematical equations for the system to be simulated.

    2. Bring the equations into a convenient form for solution on an analog computer.

    3. Develop a simplified block diagram to indicate the basic signal flow in the computer program.

    4. Perform amplitude and time scaling on the original equations.

    5. Develop final computer diagram.

    6. Perform static check calculations.

    7. Patch the problem, monitor static check voltages, and run the problem.

SYSTEM EQUATIONS

Starting from the physical system one can derive a mathematical description using the commonly-known physical laws, experimental results and any reasonable simplifications that might appear desirable. The description will include a set of equations, coefficients values, and possibly some experimental curves relating variables in the study.

Consider now the available analog computing equipment and decide what equation modifications, if any, are necessary (for instance, linearizing wherever possible to reduce the complexity of the study, using finite differences to reduce the number of continuous independent variables in a partial differential equa-tion). Make any changes of variables or formal modifications that facilitate programming. Remember that the equations most conveniently solved on the computer are ordinary differential equations.

REARRANGING THE EQUATIONS

The system equations (usually differential equations) must now be brought to a form in which an analog computer of the general type can solve them. The original equations are rearranged to have only the term with the highest derivative on the left-hand side.

$$\frac{d^n y}{dt^n} = f\left(\frac{d^{n-1} y}{dt^{n-1}}, \frac{d^{n-2} y}{dt^{n-2}}, \ldots y, F(t)\right) \tag{15}$$

Successive integration of the highest derivative will yield all the lower derivatives on the right-hand side of the equation. For solution on the computer it is assumed that $\frac{d^n y}{dt^n}$ exists as the input to an integrator; by producing the lower derivatives by integration one can sum appropriately all terms on the right-hand side of the equation to form $\frac{d^n y}{dt^n}$. This sum of terms must therefore be fed into the integrator where $\frac{d^n y}{dt^n}$ was assumed to exist.

Example:

    Given equation:

$$a_1 \frac{d^2 y}{dt^2} + a_2 \frac{dy}{dt} + a_3 y = a_4 F(t) \tag{16}$$

    Rearranged equation:

$$\frac{d^2 y}{dt^2} = -\frac{a_2}{a_1} \frac{dy}{dt} - \frac{a_3}{a_1} y + \frac{a_4}{a_1} F(t) \tag{17}$$

Assume $\frac{d^2y}{dt^2}$ exists and integrate twice to form $\frac{dy}{dt}$ and y. Generate $F(t)$ and form the R.H.S. (right-hand side of the equation (17) by summation. Use this sum for $\frac{d^2y}{dt^2}$.

BLOCK DIAGRAM

The block diagram is a preliminary computer diagram ignoring scales, signs, gains, but showing computing elements and their interconnections. It is frequently useful for obtaining an estimate of the amount of computing equipment needed, and for producing at a later time a neat computer circuit diagram.
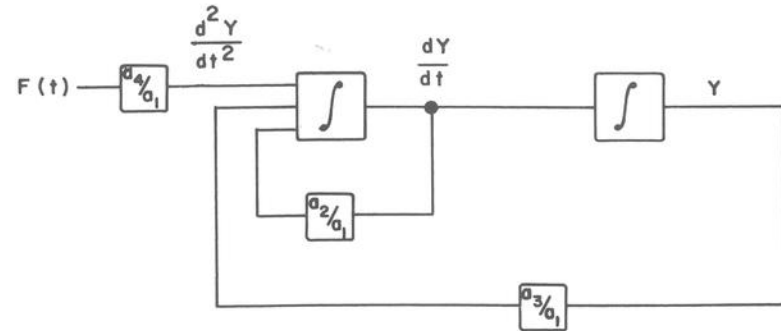
Figure 23. Block Diagram for Equation 17

SCALING

Depending on the dimensional units used, variables can have values or rates of change that range from very small to very large numbers. When preparing a problem for solution on the computer, it is necessary to arrange that voltages representing these variables will never have values larger than the allowable value of ±10 volts, nor will they change rapidly enough to exceed the frequency limitations of the computing and recording equipment.

The maximum values which quantities can assume in a particular physical system can usually be estimated from:

    a. A knowledge of the physical system under consideration.

    b. An investigation of the system equations. Steady state relations and characteristic frequencies may be employed to evaluate the ranges of the dependent variables.

AMPLITUDE SCALING

For maximum accuracy, the signal voltages representing the dependent variables and their derivatives should be as large as possible without causing the equipment to overload. The equipment is designed for signal voltages within the range of ±10 volts. The maximum values of the physical variables must therefore be scaled so that:

$$\text{(maximum value of physical variable)} \times a \leqq 10 \text{ volts} \tag{18}$$

$$\text{where } a \leqq \frac{10}{\text{max. value of physical variable}} \quad \frac{\text{volts}}{\text{physical unit}} \tag{19}$$

is called the amplitude scale factor.

The following examples demonstrate how to determine amplitude scale factors for a problem:

Physical Variable	Scale Factor a	Scaled Variable
Linear velocity $\dot{x} \leqq 100$ ft/sec	$\frac{1}{10} \frac{\text{volts}}{\text{ft/sec}}$	$[\frac{\dot{x}}{10}]$
Linear displacement $x \leqq 1500$ ft	$\frac{1}{200} \frac{\text{volts}}{\text{ft/sec}}$	$[\frac{x}{200}]$
Angular acceleration $\ddot{\theta} \leqq 1\frac{1}{2}$ rad/sec²	$5 \frac{\text{volts}}{\text{rad/sec}^2}$	$[5\ddot{\theta}]$
Temperature $T \leqq 800°$ F	$\frac{1}{100} \frac{\text{volts}}{°F}$	$[\frac{F}{100}]$

To identify scaled variables which will appear as voltages in the computer circuit, square brackets shall be used.

TIME SCALING

The ideal duration of a single run on the computer is somewhere from 15 to 60 seconds. Too long a solution time is inefficient and may introduce small errors due to integrator drift. The minimum time for a computer run is normally determined by the speed of response of the computing and recording units. If mechanically driven recording equipment is used, the frequencies occurring in the problem setup should be of the order of 1 to 2 cps. To satisfy these recommendations a change in time scale can be introduced by applying a time scale factor β, so that

$$\tau = \beta t \tag{20}$$

$$\tau = \text{machine time in seconds}$$
$$t = \text{real time in seconds}$$

Furthermore,

$$\frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = \beta\frac{dx}{d\tau} \tag{21}$$

Let us again consider an equation of the form:

$$\frac{d^2y}{dt^2} = -K_1\frac{dy}{dt} - K_2 y + K_3 F(t) \tag{22}$$

If we set $\frac{dy}{dt} = z$ we obtain:

$$\frac{dz}{dt} = -K_1 z - K_2 y + K_3 F(t) \tag{23}$$

Furthermore let it be assumed that the amplitudes of the variables were scaled as follows:

$$[a_1\frac{d^2y}{dt^2}] = [a_1\frac{dz}{dt}]; \ [a_2\frac{dy}{dt}] = [a_2 z], \ [a_3 y], \ [a_4 F(t)]$$

Taking account of the amplitude scale factors and time scaling the equations to be programmed on the com-

puter become:

$$\frac{\beta \left[a_1 \dfrac{dz}{d\tau} \right]}{a_1} = - K_1 \frac{[a_2 z]}{a_2} - K_2 \frac{[a_3 y]}{a_3} + K_3 \frac{\left[a_4 F\left(\dfrac{\tau}{\beta} \right) \right]}{a_4} \tag{24}$$

or

$$\left[a_1 \frac{dz}{d\tau} \right] = \frac{1}{\beta} \frac{a_1 K_1}{a_2} [a_2 z] - \frac{1}{\beta} \frac{a_1 K_2}{a_3} [a_3 y] + \frac{1}{\beta} \frac{a_1 K_3}{a_4} \left[a_4 F\left(\frac{\tau}{\beta}\right) \right] \tag{25a}$$

and

$$\left[a_2 \frac{dy}{d\tau} \right] = \frac{1}{\beta} [a_2 z] \tag{25b}$$

COMPUTER DIAGRAM

The resulting computer circuit, appropriately scaled, is shown in *Figure 24.*

Note in the circuit shown in *Figure 24* that every input to an integrator is divided by the time scale factor β. It is possible, in fact, to establish the following general rule:

For a change in time scale every input to an integrator must be multiplied by $\frac{1}{\beta}$.

$$\frac{1}{\beta} < 1 \quad \text{for slowing down a problem.}$$

$$\frac{1}{\beta} > 1 \quad \text{for speeding up a problem.}$$

STATIC CHECK CALCULATIONS

The static check is a procedure ensuring that the problem set on the machine corresponds to the original equations. A set of arbitrary check values are chosen for the variables and their $(n-1)$ derivatives and the value of the nth derivative (left side of the rearranged equation) is calculated. When the problem is set up on the computer, initial conditions corresponding to these check values are imposed, the voltage for the
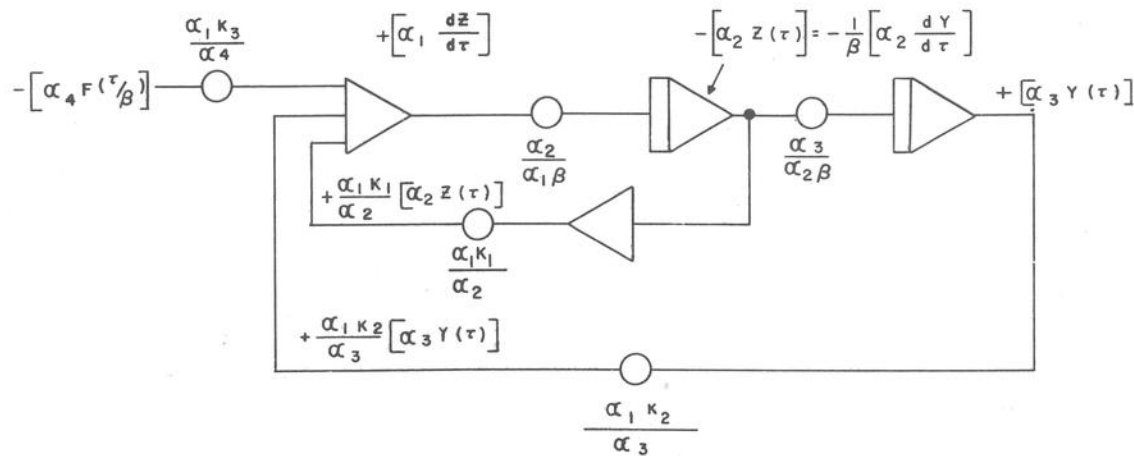
Figure 24. Computer Circuit Representing Equation 25

n^{th} derivative is measured and compared with the previously calculated scaled value. Various other voltages appearing at outputs of amplifiers should also be calculated and measured.

EXAMPLE OF AMPLITUDE AND TIME SCALING

a. Scaling Initial Conditions.

If an integrator delivers, say $- [5x]$ and the initial condition is $x \, (t = 0) = X_0 = 1.2$, this is realized by introducing a constant voltage at the IC-terminal of the integrator, taking account of the sign reversal between IC input and integrator output.

The constant voltage is obtained from a potentiometer fed by ± 10 volts reference. The output of the potentiometer should be $(5) \, (X_0) = (5) \, (1.2) = 6$ volts, and the potentiometer setting, considering the 10 volt input, is therefore

$$\frac{5 x_0}{10} = 0.600 \tag{26}$$

Since x_0 is positive and the output of the integrator is defined as $- [5x]$, we must take the $+$ reference voltage.
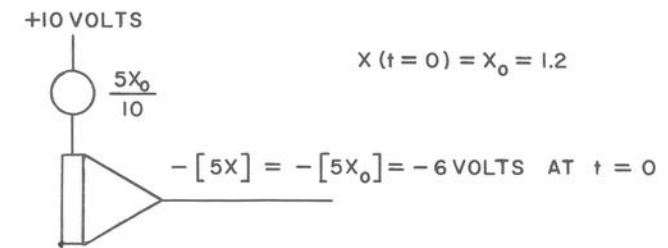
Figure 25. Scaling Initial Conditions

b. Scaling Variable Inputs.

If it is required to generate a function

$$y = g(t) \tag{27}$$

as an input into the computer system, we scale the function as follows:

First we must find the maximum value of y and amplitude scale accordingly, say $[20y]$. The time scale relation is $\tau = \beta t$. Therefore:

$$[20y] = 20 g \left(\frac{\tau}{\beta} \right) = g^1 (\tau) \tag{28}$$

The computer circuit or external equipment producing $g(t)$ must be arranged to deliver after τ seconds running time an output of $g^1(\tau)$ volts.

c. Explicit Time Dependence

If the original system is of the form

$$\frac{dX_i}{dt} = f_i \, (X_i \dots X_n, t) \tag{29}$$

with t appearing explicitly in the expression on the R.H.S. (right-hand side) of the equation, we must treat t as a quantity to be scaled as a voltage. Wherever t appears on the R.H.S. replace it by z, and introduce a second equation

$$z = t \text{ or } \frac{dz}{dt} = 1 \tag{30}$$

z has to be scaled, and this means that we have an upper boundary t_{max} which, once chosen, cannot be exceeded. t_{max} is the number of units of t over which the solution is needed.

If $t_{max} = 50$, we have $z \leqq 50$ and introduce the bracket $[\frac{z}{5}]$.

This has nothing to do with the time scaling $r = \beta t$. It is the voltage scaling of the variable z. The choice of β determines how fast the solution will reach t_{max}. Including time scaling the voltage representing z is generated by solving the equation

$$\frac{d(\frac{z}{5})}{d\tau} = \frac{d(\frac{t}{5})}{d\tau} = \frac{1}{5\beta} = \frac{1}{50\beta}[10] \tag{31}$$

Where $\frac{1}{50\beta}$ is the coefficient of the potentiometer giving the input to the time base integrator. *(Figure 26)*

The voltage from this integrator is used whenever a term involving z (originally t) is to be formed in the expression on the R.H.S. of the programmed equation.
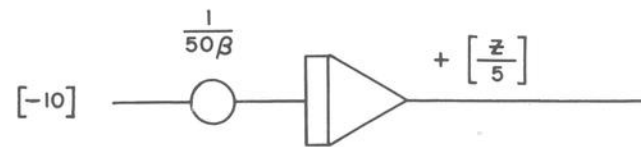
Figure 26. Time Base Integrator

ILLUSTRATIVE EXAMPLES

1. A Simple Linear Problem.

The dynamic behavior of the simple spring-mass-damping system as shown in *Figure 27* shall be investigated on the analog computer.

x is the displacement of the mass measured from the position where the spring exerts a force mg to balance the gravitational force. The parameters m, C, and K are constants. Initially the mass will be assumed to be released from rest at $x = X_0$. The displacement x, the velocity $\dot{x}$ and the acceleration $\ddot{x}$ shall be recorded on a multichannel recorder.

System equation:

$$m\frac{d^2x}{dt^2} + C\frac{dx}{dt} + Kx = 0 \tag{32}$$

Rearranged equation:

$$\frac{d^2x}{dt^2} = -\frac{C}{m}\frac{dx}{dt} - \frac{K}{m}x \tag{33}$$

AMPLITUDE SCALING

In this problem an estimate of the magnitudes of the variables can be made by employing the undamped

GIVEN DATA

$$-0.18 \text{ ft.} \leq X_{MAX} \leq 0.18 \text{ ft}$$

$$K = 420 \text{ LBS/FT}$$
$$m = 0.08 \text{ SLUGS}$$
$$C = 2.0 \frac{\text{LBS}}{\text{FT/SEC}}$$
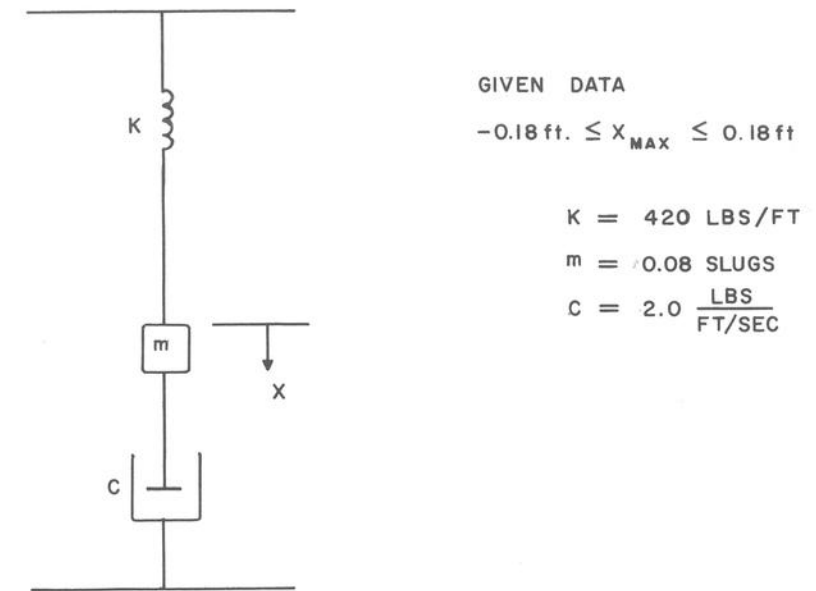
Figure 27. Spring-mass-damping System
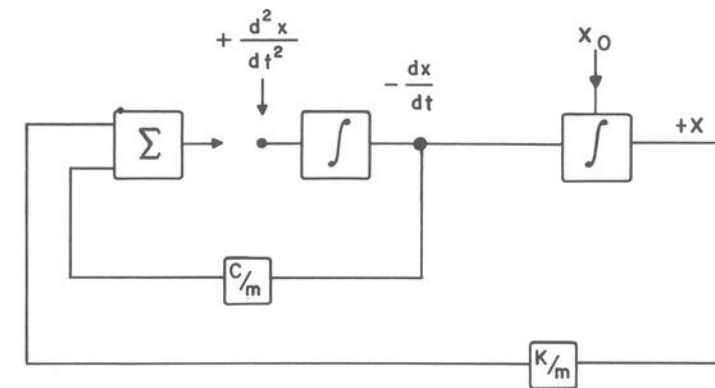
Figure 28. Block Diagram for Equation 33

natural frequency of the system,

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{420}{0.08}} \approx 73 \text{ rad/sec} \tag{34}$$

Assuming a solution of $x = \sin \omega t$ $\tag{35a}$

we obtain: $\frac{dx}{dt} = \omega \cos \omega t$ $\tag{35b}$

$$\frac{d^2x}{dt^2} = -\omega^2 \sin \omega t \qquad (35c)$$

Thus, for a maximum amplitude $x_{max} = 0.18$ ft we can estimate for a damped system the maximum values for the velocity

$$\frac{dx}{dt} < 0.18 \ \omega = (0.18)(73) \approx 13 \ \text{ft/sec}$$

and the acceleration

$$\frac{d^2x}{dt^2} < 0.18 \ \omega^2 = (0.18)(73)^2 \approx 950 \ \text{ft/sec}$$

To represent these variables by voltages which may not exceed ± 10 volts we have to apply amplitude scale factors so that the scaled variables become:

$$[50x]; \ [\frac{\dot{x}}{2}]; \ [\frac{\ddot{x}}{100}]$$

TIME SCALING

From the original equation we find that the maximum system frequency can be approximately 12 cps. In view of the recording equipment used a time scale of $\beta = 10$ seems a good choice. Every input to an integrator must therefore be multiplied by $\frac{1}{\beta} = \frac{1}{10}$ which will result in a computer solution which is ten times slower than the real physical process.

We can now write the scaled equation by simply introducing the amplitude-scaled variables into the original equation. At the same time we shall make proper adjustments to keep the equation true. We may neglect the time scale factors β in the process of writing scaled equations, as long as we remember to add it in the computer circuit.

Scaled equation:

$$[\frac{x}{100}] = \frac{1}{100}\left\{ -\frac{C}{m}(2)[\frac{\dot{x}}{2}] - \frac{K}{m}\frac{[50x]}{(50)} \right\} \qquad (36a)$$

$$[\frac{x}{100}] = -\frac{2C}{100m}[\frac{\dot{x}}{2}] - \frac{K}{5000m}[50x] \qquad (36b)$$

FINAL COMPUTER DIAGRAM:

From the scaled equation and with the help of the block diagram we can now develop the final computer circuit shown in *Figure 29*.

Note that the settings of all potentiometers in the above circuit are less than unity. This is accomplished by dividing the pot settings dictated by the scaled equation by an appropriate constant. To maintain the equation these constants were compensated for by using an amplifier gain of the same magnitude.

STATIC CHECK CALCULATIONS

The original system equation is:

$$\frac{d^2x}{dt^2} = -\frac{C}{m}\frac{dx}{dt} - \frac{K}{m}x \qquad (37)$$
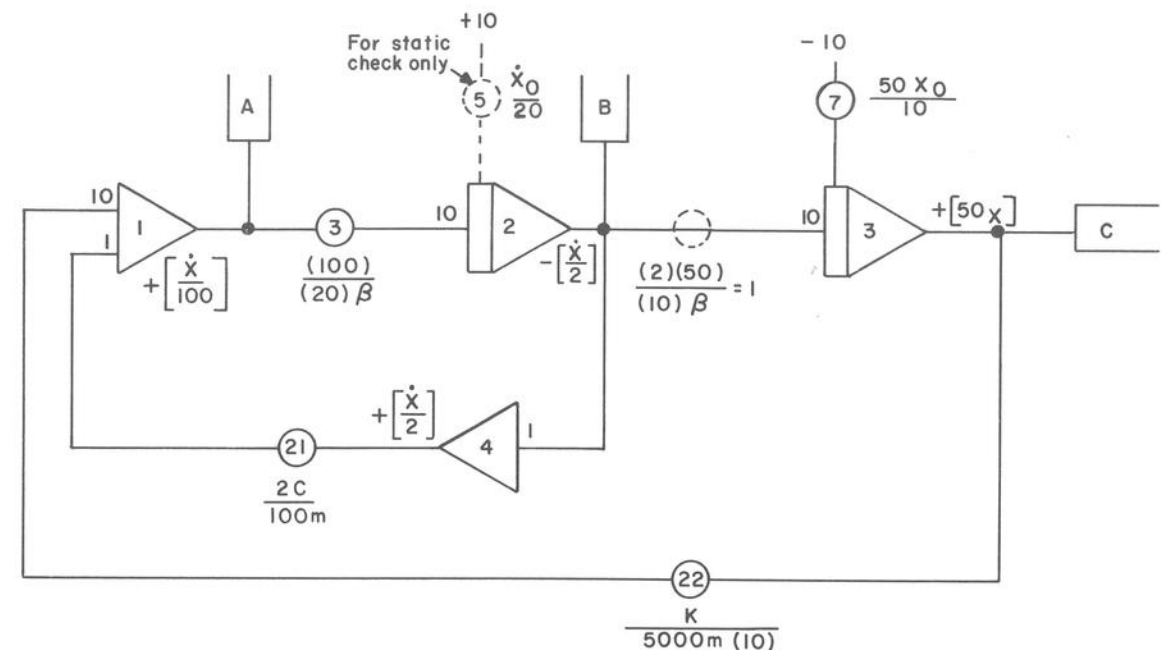
Figure 29. *Computer Circuit for Spring-mass-damping System*

Assume for the purposes of a static check that: $\frac{dx}{dt} = 8; \ x = 0.1$

$$\frac{d^2x}{dt^2} = -\frac{2}{0.08}(8) - \frac{420}{0.08}(0.1) = -725 \qquad (38)$$

If the problem was correctly programmed and patched, the following voltages should exist in the computer circuit:

$$+[\frac{\ddot{x}}{100}] = \frac{-725}{100} = -7.25 \text{ volts output of amplifier 1}$$

$$-[\frac{x}{2}] = -\frac{8}{2} = -4.00 \text{ volts initial condition on amplifier 2}$$

$$+[50x] = (50)(0.1) = +5.00 \text{ volts initial condition on amplifier 3}$$

$$+[\frac{\dot{x}}{2}] = +4.00 \text{ volts output of amplifier 4}$$

$$\frac{2C}{100m}[\frac{\dot{x}}{2}] = +2.00 \text{ volts output of potentiometer 21}$$

$$\frac{K}{5 \times 10^4 m}[50x] = +0.525 \text{ volts output of potentiometer 22.}$$

The Voltmeter may be used to monitor these voltages. After the static check has been completed the computer is ready for operation to investigate the behavior of the mass-spring system under various initial conditions or changes in the values of m, C and K.

SOLUTION OF SIMULTANEOUS DIFFERENTIAL EQUATIONS

The programming of a set of simultaneous differential equations presents no more basic difficulties than

the solution of a single equation. The following problem serves as an illustration.

AN AUTOMOBILE SUSPENSION SYSTEM

Assume the task is to investigate the response of an automobile suspension system for selected disturbances. The system response to these disturbances for various values of the system design parameters shall be obtained.
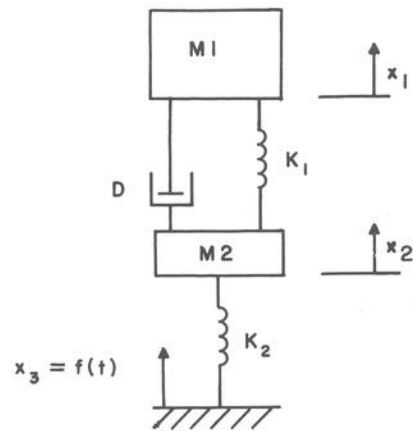
Physical Constants	Ranges of Variables
$M_1 = 25$ slugs	$x_1 \leqq 6$ in.
$M_2 = 2$ slugs	$x_2 \leqq 12$ in.
$K_1 = 1000$ lb/ft	$x_3 \leqq 6$ in.
$K_2 = 4500$ lb/ft	
$D = $ Variable	(See Figure 30.)

$M_1 = $ One quarter of mass of automobile

$M_2 = $ Mass of one wheel and half of the axle

$K_1 = $ Constant of main auto spring

$K_2 = $ Spring constant of tire (assumed linear)

$D = $ Shock absorber damping constant

$x_1 = $ Displacement of auto body

$x_2 = $ Displacement of wheel

$x_3 = $ Roadway profile displacement

Figure 30. Simplified Automobile Suspension System

SYSTEM EQUATIONS

The differential equations of motion of the system are derived by equating forces acting upon the masses. Thus for M_1:

$$M_1 \frac{d^2x_1}{dt^2} + D\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right) + K_1(x_1 - x_2) = 0 \tag{39}$$

and for M_2:

$$M_2 \frac{d^2x_2}{dt^2} + D\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right) + K_1(x_2 - x_1) + K_2(x_2 - x_3) = 0 \tag{40}$$

Estimate of maximum values:

The undamped natural frequency of M_1 is

$$\omega_{n1} = \sqrt{\frac{K_1}{M_1}} = \sqrt{\frac{1000}{25}} = 6.32 \text{ rad/sec} \tag{41}$$

For the mass M_2 we obtain:

$$\omega_{n2} = \sqrt{\frac{K_1 + K_2}{M_2}} = \sqrt{\frac{5500}{2}} = 52.4 \text{ rad/sec} \tag{42}$$

Reasonable estimates for the maximum values of the variables and their derivatives are:

$$x_{1max} = 0.5 \, ft \quad x_{2max} = 1 \, ft \quad x_{3max} = 1 \, ft$$

$$\dot{x}_{1max} = (x_{1max})\omega_{n1} = (0.5)(6.32) \approx 3.2 \, ft/sec$$

$$\dot{x}_{2max} = (x_{2max})\omega_{n2} = (1)(52.4) \approx 50 \, ft/sec$$

$$\ddot{x}_{1max} = (\dot{x}_{1max})\omega_{n1} \approx 20 \, ft/sec^2$$

$$\ddot{x}_{2max} = (\dot{x}_{2max})\omega_{n2} \approx 2500 \, ft/sec^2$$

We can therefore scale the variables so that the following voltages will appear in the computer circuit:

$$[20x_1] \quad [10x_2] \quad [10x_3]$$

$$[2\dot{x}_1] \quad [\frac{\dot{x}_2}{5}]$$

$$[\frac{\ddot{x}_1}{2}] \quad [\frac{\ddot{x}_2}{500}]$$

Solving the original equations for the highest derivatives and introducing the scaled variables yields the scaled equations:

$$[\frac{\ddot{x}_1}{2}] = \frac{5D}{2M_1}[\frac{\dot{x}_2}{5}] - \frac{D}{4M_1}[2\dot{x}_1] + \frac{K_1}{20M_1}[10x_2] - \frac{K_1}{40M_1}[20x_1]$$

$$[\frac{\ddot{x}_2}{500}] = \frac{D}{1000M_2}[2\dot{x}_1] - \frac{D}{100M_2}[\frac{\dot{x}_2}{5}] + \frac{K_1}{10^4 M_2}[20x_1] - \frac{K_1 + K_2}{5000M_2}[10x_2] + \frac{K_2}{5000M_2}[10x_3] \tag{43}$$

The coefficients in these equations can each be separated into two parts, -- an attenuator setting and an amplifier gain. Assuming a range of values of D up to 100 we have on rewriting the equations:

$$[\frac{\ddot{x}_1}{2}] = 10 \, (\frac{D}{4M_1})[\frac{\dot{x}_2}{5}] - (\frac{D}{4M_1})[2\dot{x}_1] + 10(\frac{K_1}{200M_1})[10x_2] - (\frac{K_1}{40M_1})[20x_1]$$

$$[\frac{\ddot{x}_2}{500}] = (\frac{D}{1000M_2})[2\dot{x}_1] - (\frac{D}{100M_2})[\frac{\dot{x}_2}{5}] + (\frac{K_1}{10^4 M_2})[20x_1] - (\frac{K_1 + K_2}{5000M_2})[10x_2] + (\frac{K_2}{5000M_2})[10x_3] \tag{44}$$

From the scaled equations we can now develop the computer circuit shown in Figure 31. An attenuator placed immediately before an integrator must produce with the gain shown on that integrator the necessary change in scale factor across the integrator, e.g. a setting of $\frac{5}{\beta}$ followed by a gain 10 causes $-[\frac{\dot{x}_2}{5}]$ to be integrated to $+[10x_2]$. The factor $\frac{1}{\beta}$ permits the inclusion of a time scale change. In this problem $\beta = 10$ appears appropriate, slowing down the solution by a factor of 10 to 1.

The disturbances $x_3 = f(t)$ may be fed to the computer from a recording of road noise, or the voltage $[10x_3]$ may be a suitable artificial disturbance produced on the computer -- a step, ramp or sinusoidal time function.
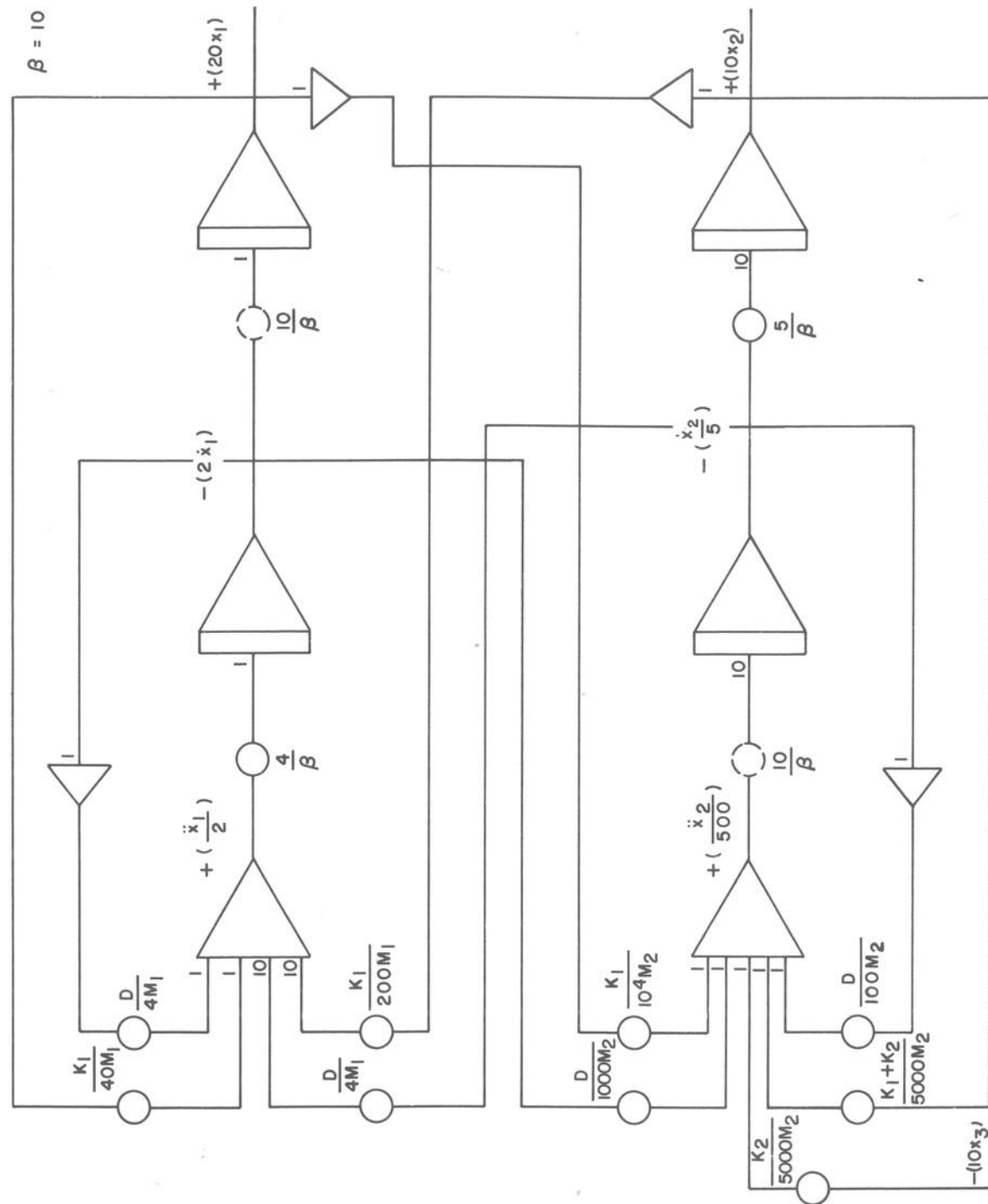
Figure 31. Computer Circuit for Automobile Suspension System

The voltages representing x_1 and x_2 can be recorded on a strip-chart recorder or on a VARIPLOTTER. Typical curves of time behavior for an input x_3 chosen as a 35 millisecond pulse are shown in *Figure 32*. The effect of changing the value of D is obvious.

A NON-LINEAR PROBLEM

The problem is to compute a curve of radial velocity of a particle vs radial distance from the center of a disc used in a spray drier. The results of the computation shall be used to select the design parameters of a drier for a particular duty.

The differential equation which describes the velocity of the particle is:

$$V\frac{dV}{dr} + AV^3 - Br = 0 \tag{45}$$

V = Radial velocity [in/sec]

r = Radial distance from center of disc [inches]

$A, B,$ = Constants which in the case of laminar flow depend upon the angular velocity of the disc, volumetric feed rate, vane height and the density and viscosity of the liquid. For turbulent flow the same equation holds, with different values for constant A.

The velocity will be recorded as a function of the radius r on a VARIPLOTTER.

Considering the original equation we must replace the independent variable r by time t, since the computer integrates with respect to time only. Let us therefore introduce

$$r = kt \tag{46}$$

For $k = 1$ we obtain:

$$\frac{dV}{dt} = \frac{Bt}{V} - AV^2 \tag{47}$$

The constants A and B will be varied for several runs on the computer. Their mean values are:

$$A = 1.66 \times 10^{-4}$$
$$B = 3.6 \times 10^7$$

The velocity V will have a range of:

$$V = 0 \ \dots \ 6000 \ \text{in/sec}$$

for a range of radius $r = 0 \dots 5$ inches. We can therefore scale the variables as:

$$[2t]; \ [10^{-3}V]$$

With the scaled variables we obtain the scaled equation as follows:

Introducing amplitude scale factors and maintaining the terms of the equation consistent:

$$[10^{-3}\frac{dV}{dt}] = \frac{10^{-6}}{2}B\frac{[2t]}{[10^{-3}V]} - \frac{A}{10^{-3}}[10^{-3}V]^2 \tag{48}$$

Including the factor of ten which occurs in the output when doing multiplication and division (see page 15),

$$[10^{-3}\frac{dV}{dt}] = \frac{10^{-7}}{2}B\frac{[10][2t]}{[10^{-3}V]} - \frac{A}{10^{-4}}\frac{[10^{-3}V]^2}{[10]} \tag{49}$$

(a)

(b)

(c)
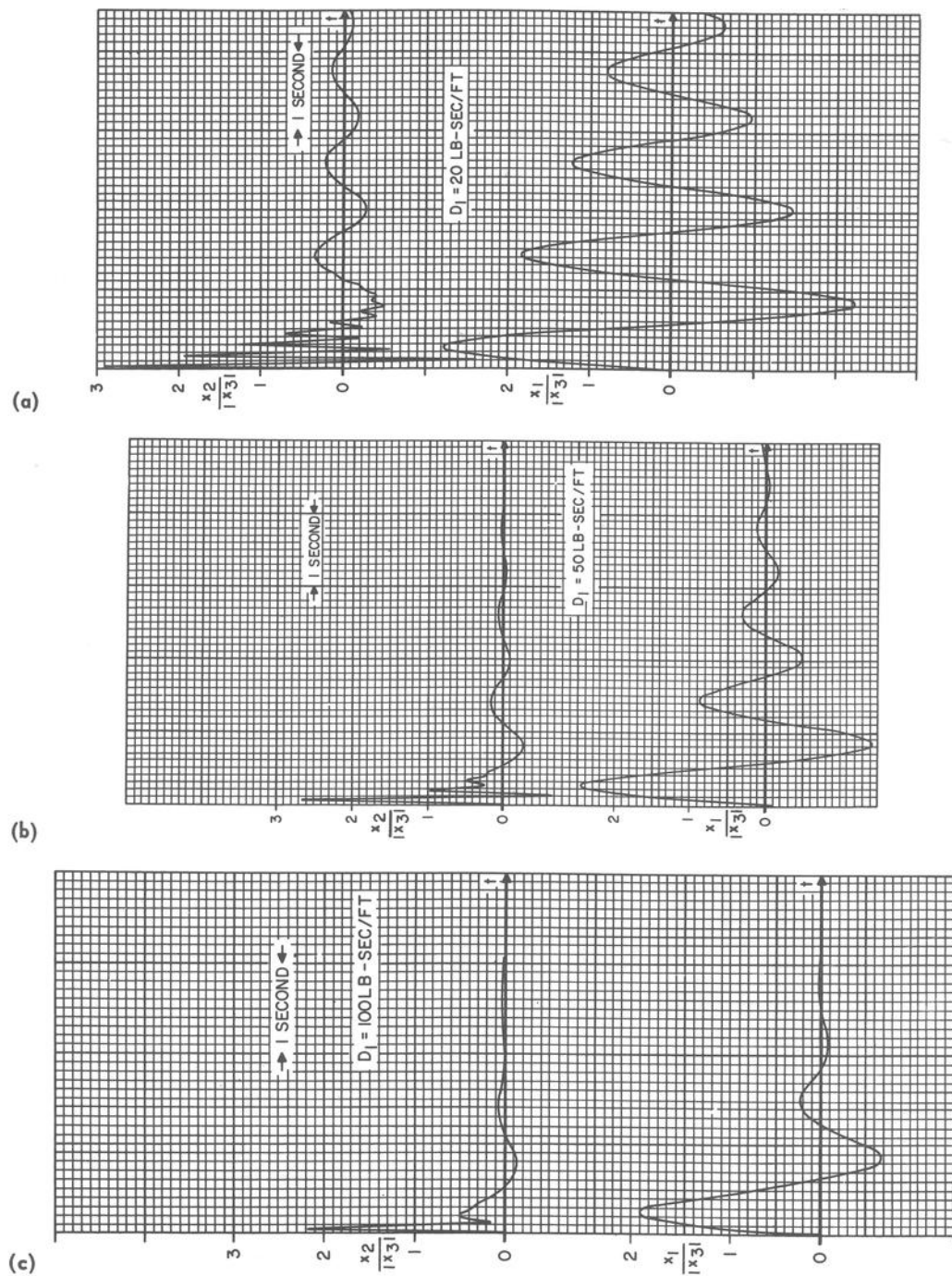
Figure 32. Displacement of Auto Body $(\frac{X_1}{|X_3|})$ and Wheel $(\frac{X_2}{|X_3|})$ for a Deflection in Road Profile (X_3 Is a

Square Pulse of 35 Milliseconds Duration)

Dividing the coefficients where necessary into two parts, one less than unity, the other a power of 10, so that they can be obtained using attenuators and amplifier gains,

$$[10^{-3}\frac{dV}{dt}] = 10 \ (\frac{B}{2 \cdot 10^8}) \ \frac{[10][2t]}{[10^{-3}V]} - 10 \ (1000A)\frac{[10^{-3}V]^2}{[10]} \tag{50}$$

From this scaled equation we see that we have to generate a voltage equal to $2t$. This is achieved by feeding an integrator with a voltage of constant value -2 volts.

The computer circuit can now be developed from the scaled equation and is shown in *Figure 33*.

This circuit has no unusually large amplifier gains nor unusually small attenuator settings. Thus a time scale change appears unnecessary. Indeed the full solution time, as we know already, is five seconds, and this might be perfectly reasonable and not give rates of voltage change within the circuit which are inappropriate for the plotting equipment to be used. If a slower solution is found to be desirable, then it can be achieved by the accepted method of reducing the gains associated with each integration. The gain change is made either by resetting a potentiometer or changing the input resistor to the amplifier, whichever is the more convenient. For example, to double the time of solution the settings of attenuators 3, 4, and 8 can all be divided by $\beta = 2$.
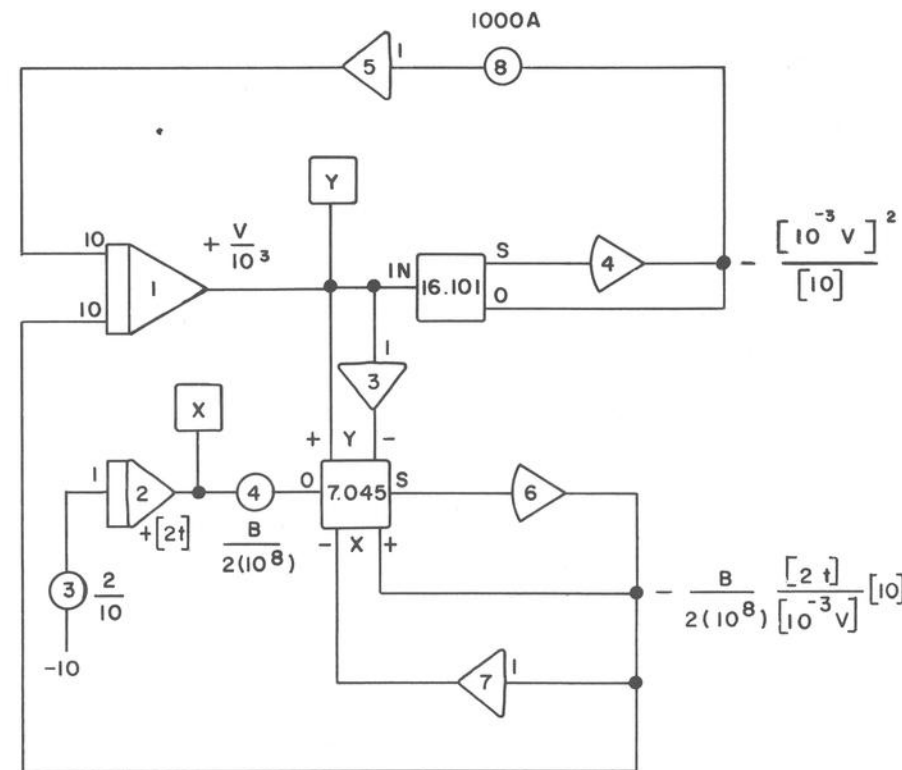
Figure 33. Computer Circuit Representing Equation 50

To convert the voltage at the output of integrator 2 into inches we must remember that $t = r$ and thus 2 volts represents 1 inch. The maximum output of the integrator will be 10 volts for $r = 5$ inches. The maximum output of the velocity integrator will be 6 volts for $V = 6000$ in/sec. We shall therefore use an arm scale (x) on the VARIPLOTTER of 2 volts/inch, while the pen scale (y) will be 1 volt/inch.

The calculation of the static check voltages should be performed similarly to the procedure indicated previously.

Typical curves obtained from the computer for several values of B with A held constant and vice versa are shown in *Figure 34*.
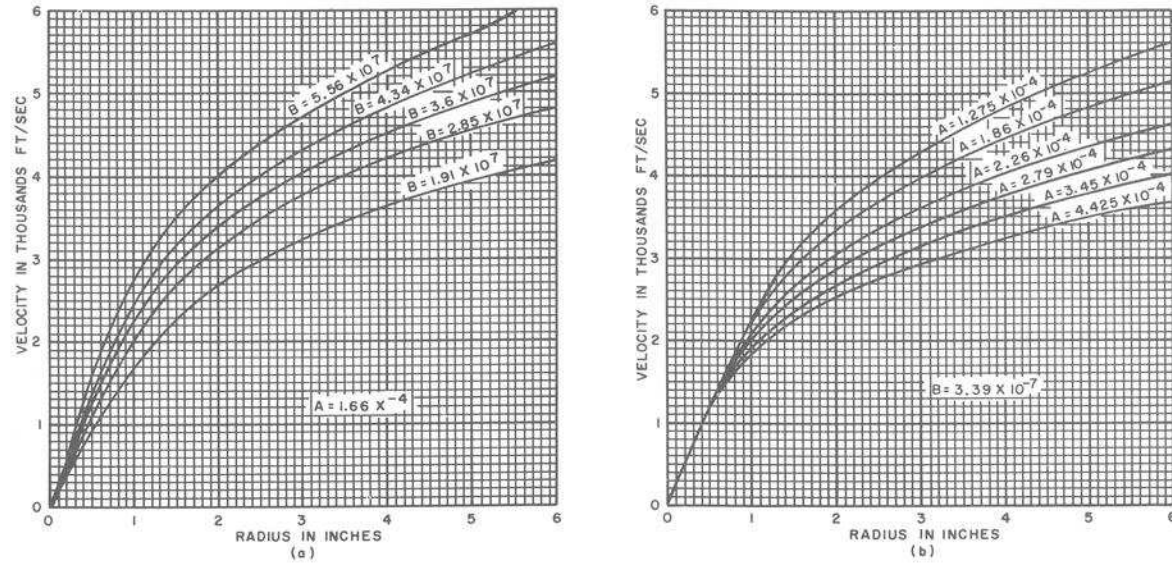
Figure 34. Plot of Radial Velocity as a Function of Drier Radius

GENERATION OF ANALYTIC FUNCTIONS

The following is a summary of some typical computer circuits for the generation of analytic functions sometimes required in the investigation of physical problems.

1. Generation of $y = e^{-\lambda t}$ (51)

$$\frac{dy}{dt} = -\lambda e^{-\lambda t} = -\lambda y \tag{52}$$

(See Figure 35.)

2. Generation of $\sin \omega t$ and $\cos \omega t$

$$\frac{d}{dt} \sin \omega t = \omega \cos \omega t \tag{53a}$$

$$\frac{d}{dt} \cos \omega t = -\omega \sin \omega t \tag{53b}$$

$$\sin \omega t = \omega \int \cos \omega t\, dt \tag{53c}$$
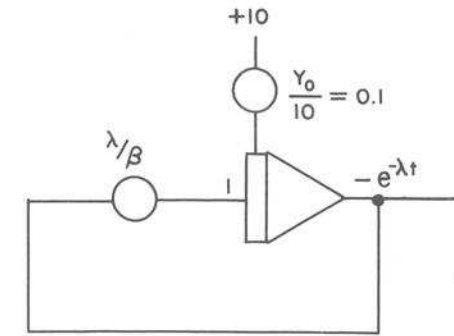
Figure 35. Generation of $y = C^{-\lambda t}$

$$\cos \omega t = -\omega \int \sin \omega t\, dt \tag{53d}$$

(See Figure 36.)

3. Generation of $y = e^{-x}$ where x and y are both dependent problem variables.

$$\frac{dy}{dt} = -\frac{dx}{dt} e^{-x} \tag{54}$$

Here it is assumed that $\frac{dx}{dt} = \dot{x}$ is available as the input function to this computer circuit.

(See Figure 37.)

4. Generation of $y = ln\ (x + a)$

$$\frac{dy}{dt} = \frac{1}{x + a} \left(\frac{dx}{dt} \right) \tag{55}$$

(See Figure 38.)

CURVE FOLLOWER 2.040

The VARIPLOTTER, with the addition of a curve follower head, may be used to produce an arbitrary or non-analytic function of an input variable. The function is drawn on graph paper with a special conducting ink. One end of the curve is then connected to the output of a 470 kc oscillator and the other end is grounded. The curve (conductor) will generate a magnetic field which on one side of the curve is
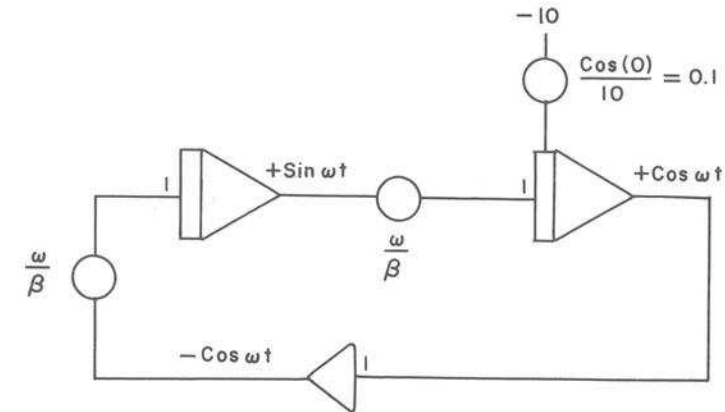
Figure 36. Generation of $\sin \omega t$ and $\cos \omega t$

180° out of phase with the field on the other side of the curve. (A curve follower head is mounted in place of the pen of the VARIPLOTTER.) The curve follower head will sense an excursion from the curve and will supply an error voltage to the pen servo-system in the VARIPLOTTER, which will return the curve follower to the center of the line of the curve. The arm of the plotter is driven by an external voltage x. The output voltage corresponding to $y = f(x)$ is obtained from the wiper of the pen follow-up potentiometer, which is not otherwise required in this use of the VARIPLOTTER. A voltage z may be applied to one end of this potentiometer, with the other end grounded. The output voltage at the wiper of the potentiometer will then be $(z) = f(x)$. For further details and instructions about the connections of the
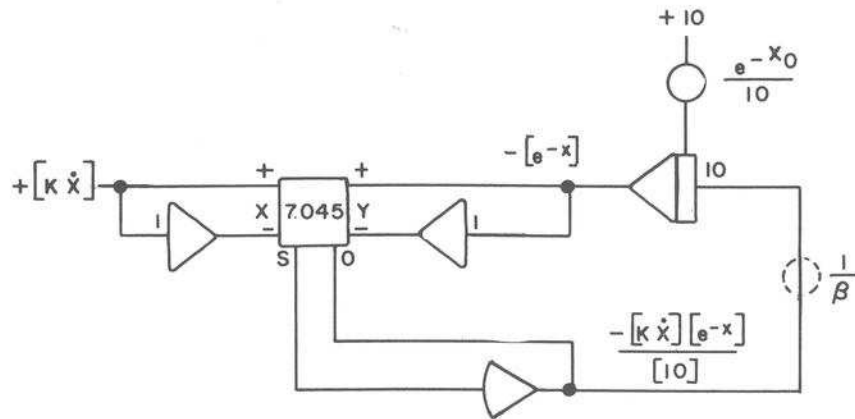
Figure 37.   Generation of $y = e^{-x}$
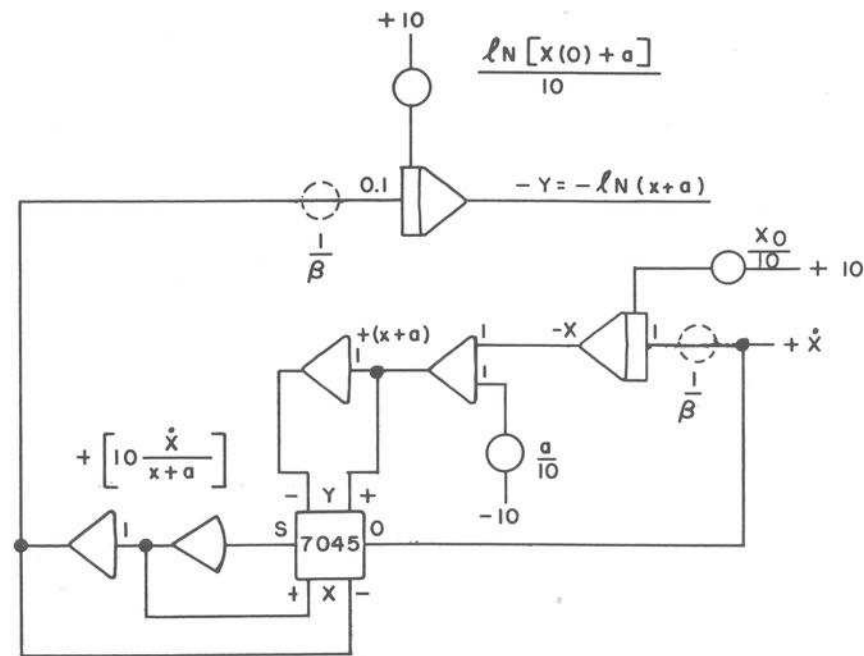
Figure 38.   Generation of $y = ln\ (x + a)$

VARIPLOTTER and the curve follower the reader is referred to the basic 1100 VARIPLOTTER manual and the Function Generator Model 2.040 manual.

EXAMPLE   Consider a spring-mass-damping system similar to the one discussed earlier. Let us assume that the damping coefficient C is now a function of the displacement x. The equation describing the system then becomes :

$$m\ddot{x} + f(x)\dot{x} + kx = 0 \qquad (56)$$

where $C = f(x)$ is given in form of a graph. *(Figure 39.)*
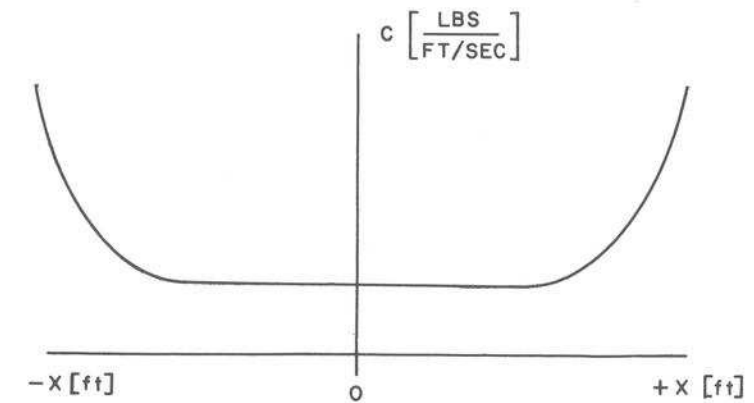
Figure 39.   Graph of Damping Coefficient C

In the investigation of this system one would prepare the graph for the VARIPLOTTER and curve follower as outlined above, and the computer circuit would have the following form (scale factors have been neglected). *See Figure 40.*

Another device, BIVAR (bi-varient function generator), may be used to generate functions of two independent variables.
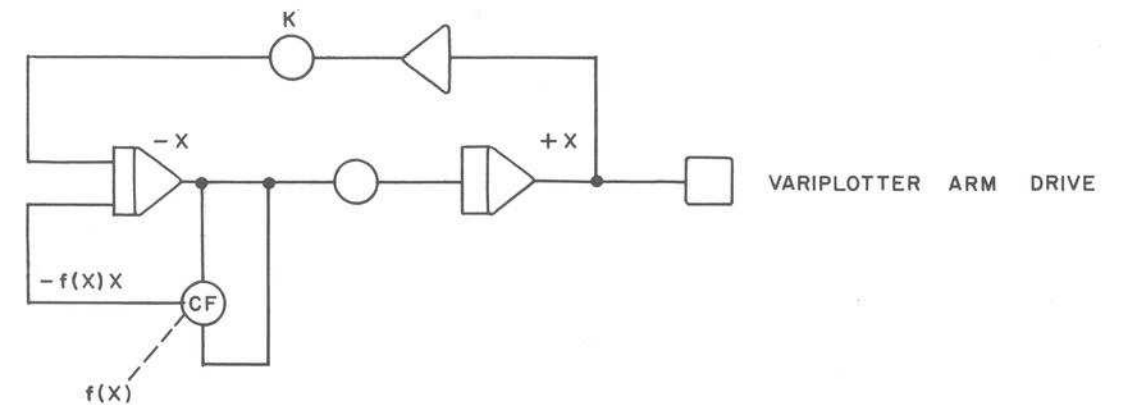
Figure 40.   Function Generation with VARIPLOTTER and Curve Follower

TRANSFER FUNCTIONS

The engineer often writes differential equations describing a system in a certain operational form. This engineering procedure uses "Transfer Functions" which are usually defined in terms of "Time Constants" These terms may be illustrated by the following simple example:

Consider a capacitor connected in series with a resistor, switch and battery *(Figure 41)*. At time t after the switch is closed, the current flowing in the circuit will be i and conditions are described by the equation:

$$E - Ri - \frac{1}{C}\int i \, dt = 0 \tag{57}$$

which may be readily solved by several methods. It is customary to use an operational notation in such equations, and one may write

$$\int [\ ] \, dt = \frac{[\ ]}{p} \tag{58a}$$

and

$$\frac{d[\ ]}{dt} = p[\ ] \tag{58b}$$

Equation 57 may be written in an operational form

$$E - Ri - \frac{1}{C}\frac{i}{p} = 0 \tag{59a}$$
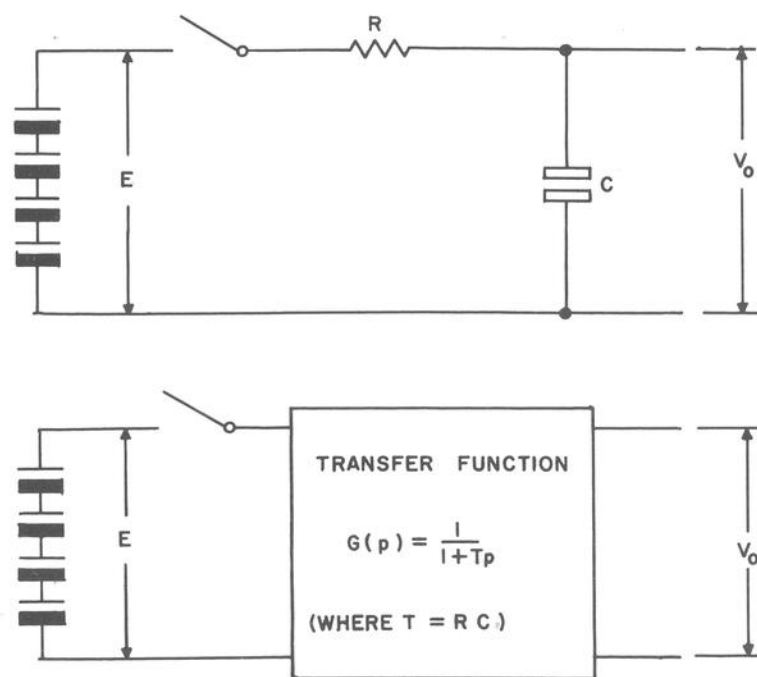
Figure 41. Simple Circuit and its Transfer Function

Whence, by treating the operator p as though it were an algebraic quantity:

$$i = (E)\left(\frac{Cp}{1 + RCp}\right) \tag{59b}$$

the output voltage $V_0 = \frac{i}{Cp}$

so that

$$V_0 = (E)\left(\frac{1}{1 + RCp}\right) \tag{60}$$

The transfer function of the circuit may be defined as the ratio of the output quantity to the input quantity. In this simple case, where the quantities are expressed in the same units, the transfer function $G(p)$ is given by

$$G(p) = \frac{V_0}{E} = \frac{1}{1 + RCp} \tag{61a}$$

The quantity RC has dimensions of time and is termed the time constant τ (The operator p has dimensions $\frac{1}{time}$)

One may write in this case:

$$G(p) = \frac{1}{1 + \tau p} \tag{61b}$$

and this function is sufficient to describe the voltage characteristics of the circuit.

If the transfer functions of a system are known or obtainable, one may readily simulate the system on an electronic analog computer.

The computer circuit for the above example is obtained as follows:

From equation: $\quad V_0 + \tau p \, V_0 = E \tag{62}$

Solving for the highest derivative:

$$p V_0 = \frac{E}{\tau} - \frac{V_0}{\tau} \tag{63}$$

Assuming that the highest derivative exists at the input to an integrator we obtain the computer circuit shown in *Figure 42*.

Occasionally the engineer will be confronted by a transfer function representing a system, which is described by a polynomial in p, divided by a polynomial in p, such as:

$$\frac{E_0}{E_i} = \frac{K_1 p^2 + K_2 p + K_3}{p^3 + K_4 p^2 + K_5 p + K_6} \tag{64}$$

The following approach is recommended for the setup of this problem:

As a first step expand the equation into differential equation form, giving:

$$p^3 E_0 + K_4 p^2 E_0 + K_5 p E_0 + K_6 E_0 = K_1 p^2 E_i + K_2 p E_i + K_3 E_i \tag{65}$$

One now proceeds by dividing the equation by p^{n-1}, where n equals the highest power of p. In this case the divisor will be p^2.

$$p E_0 + K_4 E_0 + K_5 E_0 \frac{1}{p} + K_6 E_0 \frac{1}{p^2} = K_1 E_i + K_2 E_i \frac{1}{p} + K_3 E_i \frac{1}{p^2} \tag{66}$$
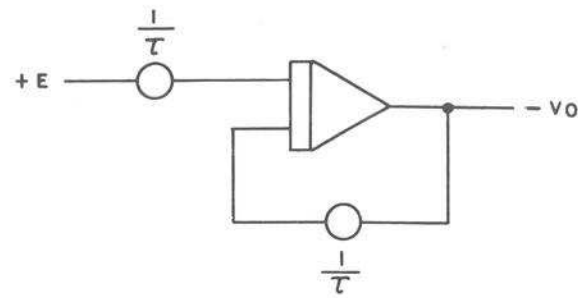
Figure 42. *Circuit Representing the Transfer Function* $G(p) = \dfrac{1}{1+\tau p}$
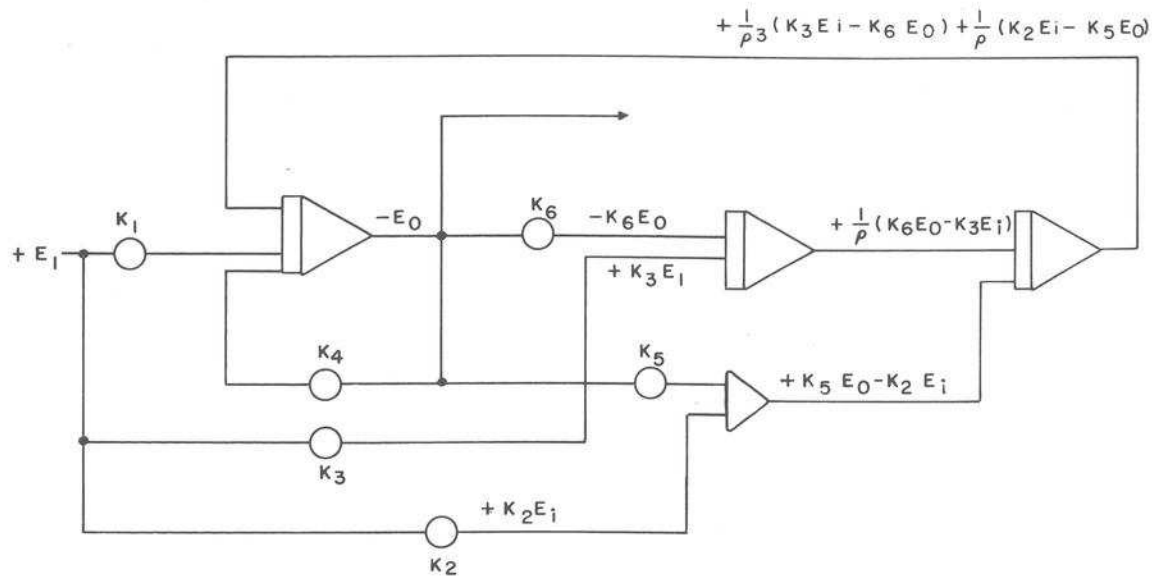

Figure 43. *Computer Circuit Representing the Transfer Function*

$$G(p) = \frac{K_1 p^2 + K_2 p + K_3}{p^3 + K_4 p^2 + K_5 p + K_6}$$

Solving the equation for the highest derivative and grouping the terms according to powers of p gives:

$$pE_0 = K_1 E_i - K_4 E_0 + \frac{1}{p^2}(K_3 E_i - K_6 E_0) + \frac{1}{p}(K_2 E_i - K_5 E_0) \qquad (67)$$

If one remembers that the notation $\frac{1}{p}$ represents an integration and if it is assumed that E_i and E_0 are known, the equation can then be programmed yielding the computer circuit shown in *Figure 43*.

III. MISCELLANEOUS ADVANCED TECHNIQUES

REPRESENTATION OF DISCONTINUITIES

A variety of problems require the simulation of non-linear effects such as displacement, limiting, hysteresis, gear backlash, etc. Most of these functions may be reasonably well represented on the analog computer using the signal voltage comparator and/or diodes in conjunction with operational amplifiers. A few basic computer circuits commonly used to represent non-linear phenomena are given below.

1. *Simple Limiting*

Desired response: $-l_1 \leqq X \leqq l_2$ *(See Figure 44.)*

The amplifier output will follow the input voltage until one of the diodes conducts. This will introduce a low impedance into the feedback path. The feedback impedance will not be zero because of the attenuator impedance $\dfrac{l}{10 + l}R_{att}$, and the forward resistance of the diode. Therefore a small gain will exist which will give a slight rise to the limited portion of the output voltage.
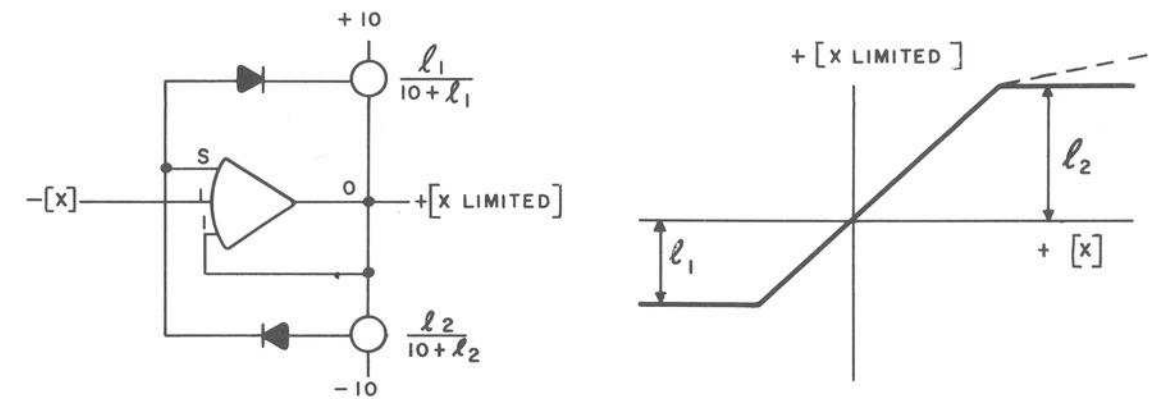
Figure 44. *Simple Limiting Circuit*

2. *Bang-Bang Circuit*

Desired response: $X_{\text{limited}} = -l_1$ for $-\infty < X < 0$

$X_{\text{limited}} = +l_2$ for $0 < X < \infty$ *(See Figure 45.)*

Note that the feedback resistor was omitted in this circuit. The output voltage of the amplifier will therefore very rapidly swing in one direction or the other depending upon the sign of the input $-[x]$ until one of the diodes conducts.

3. *Zero Limiting (Half wave rectification)*

Desired response: $X_{\text{limited}} = 0$ for $-\infty < X < 0$

$X_{\text{limited}} = X$ for $0 < X < \infty$ *(See Figure 46.)*

The diode at the output of the amplifier will introduce a high impedance when the amplifier attempts to go negative. The diode around the amplifier limits the amplifier in the negative direction for positive inputs.
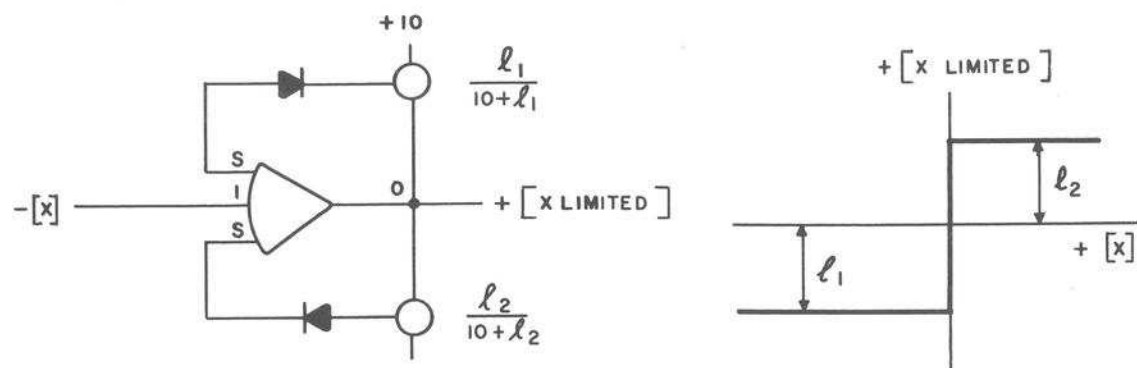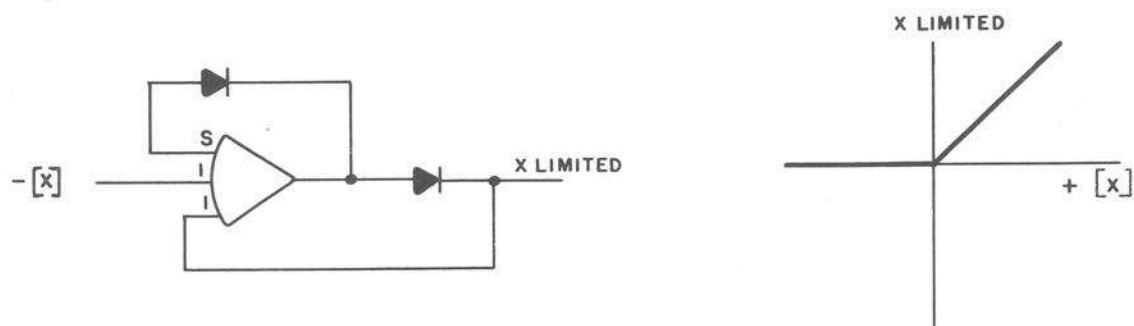
Figure 45. Bang-Bang Circuit

Figure 46. Zero Limiting Circuit

4. *An Absolute Value Circuit*

> Desired response: $X = |X|$ for all X *(See Figure 47.)*

5. *Bang-Bang Circuit Using the Signal Voltage Comparator.*   *(See Figure 48.)*

A more extensive tabulation of circuits representing common non-linearities could be easily developed. However, with the few basic circuits shown above the reader will be able to develop other circuits suitable to the requirements of a particular problem.
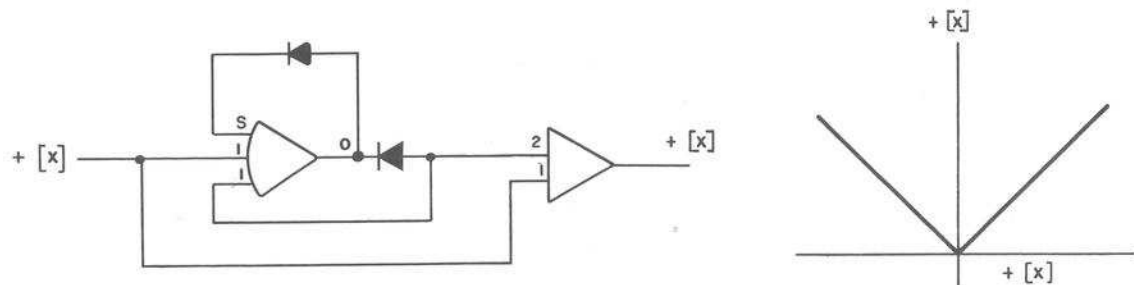
Figure 47. Absolute Value Circuit
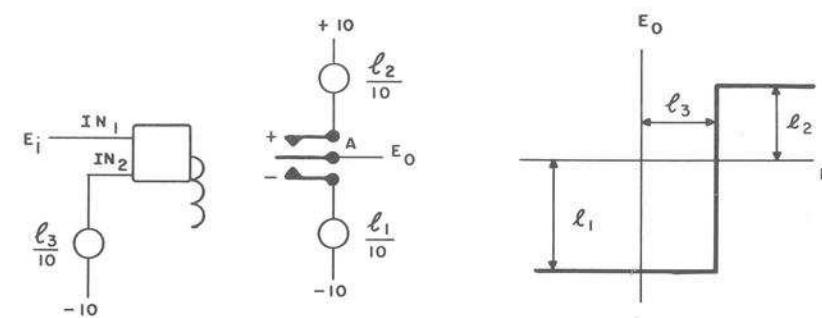
Figure 48. Bang-Bang Circuit Using the Signal Voltage Comparator

INTRODUCTION TO THE SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

Partial differential equations can be successfully solved on the analog computer. However, a mathematical manipulation of the equation is necessary for the analog computer will only integrate with respect to one variable, i.e. time.

In solving partial differential equations some method must be employed which will transform the partial differential equation into a set of ordinary differential equations. By using finite difference approximations for the derivatives with respect to all but one of the independent variables a set of finite-difference differential equations are written.

For introductory purposes let us assume we are to investigate the equation for the natural vibration of an elastic string:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T} \frac{\partial^2 y}{\partial t^2} \tag{68}$$

where ρ is the mass per unit length and T is a constant tension. For such an equation the values of y and $\frac{\partial y}{\partial t}$ should be known as a function of x for $t = 0$. These constitute the initial conditions of the problem. The boundary conditions must be known at both ends for all values of time t.

In order to reduce the partial differential equation to a set of ordinary differential equations, let us express the space derivatives in terms of finite differences. We select three points as shown in *Figure 49.* spaced Δx apart. We can then obtain the approximate values of the first derivatives at the midpoints of the two Δx intervals:

$$\frac{\partial y}{\partial x}\bigg|_{x - \frac{\Delta x}{2}} = \frac{1}{\Delta x} \ (y_x - y_{x - \Delta x}) \tag{69}$$

$$\frac{\partial y}{\partial x}\bigg|_{x + \frac{\Delta x}{2}} = \frac{1}{\Delta x} \ (y_{x + \Delta x} - y_x) \tag{70}$$

From equations 69 and 70 it is apparent that the accuracy of the approximation depends upon the size of the chosen interval Δx. Dividing the length l into segments Δx effectively means that we are solving the problem at discrete points, rather than continuously.
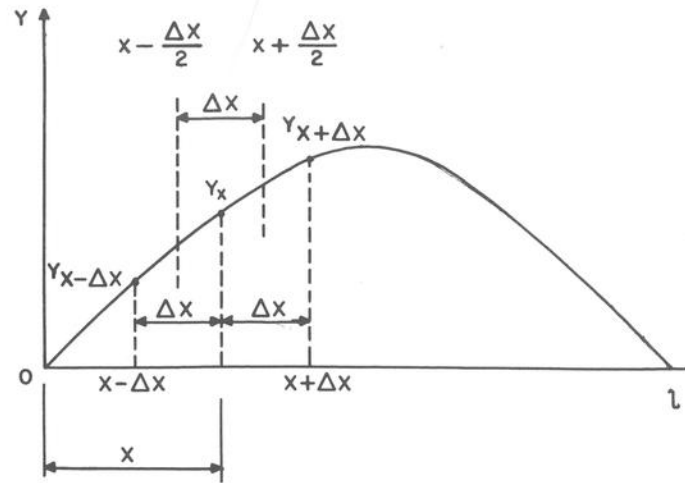
Let us now consider the second derivative for point x.

Figure 49.

It is:

$$\frac{\partial^2 y}{\partial x^2}\bigg|_x = \frac{1}{\Delta x}\left(\frac{\partial y}{\partial x}\bigg|_{x+\frac{\Delta x}{2}} - \frac{\partial y}{\partial x}\bigg|_{x-\frac{\Delta x}{2}}\right) \tag{71}$$

or

$$\frac{\partial^2 y}{\partial x^2}\bigg|_x = \frac{1}{\Delta x^2}\left(y_{x+\Delta x} - 2y_x + y_{x-\Delta x}\right) \tag{72}$$

Thus for position x we can write the equation for the natural vibration of the elastic string:

$$\frac{d^2 Y_x}{dt^2} = \frac{T}{\rho \Delta x^2}\left(y_{x+\Delta x} - 2y_x + y_{x-\Delta x}\right) \tag{73}$$

Equation 73 can be generalized for any arbitrary point n along the string so that

$$\frac{d^2 y}{dt^2}\bigg|_n = \frac{T}{\rho \Delta x^2}\left(y_{n+1} - 2y_n + y_{n-1}\right) \tag{74}$$

We shall now divide the string into five segments of length $\Delta x = l_5$ as shown in *Figure 50*.
The boundary conditions are: $y_0 = 0$ $y_5 = 0$

From equation (74) we obtain the following finite difference equations:

$$\frac{d^2 y_1}{dt^2} = \frac{25 T}{\rho l^2}\left(y_2 - 2y_1\right) \tag{75}$$

$$\frac{d^2 y_2}{dt^2} = \frac{25 T}{\rho l^2}\left(y_3 - 2y_2 + y_1\right) \tag{76}$$

$$\frac{d^2 y_3}{dt^2} = \frac{25 T}{\rho l^2}\left(y_4 - 2y_3 + y_2\right) \tag{77}$$

THE BOUNDARY CONDITIONS
ARE: $Y_0 = 0$

$Y_5 = 0$
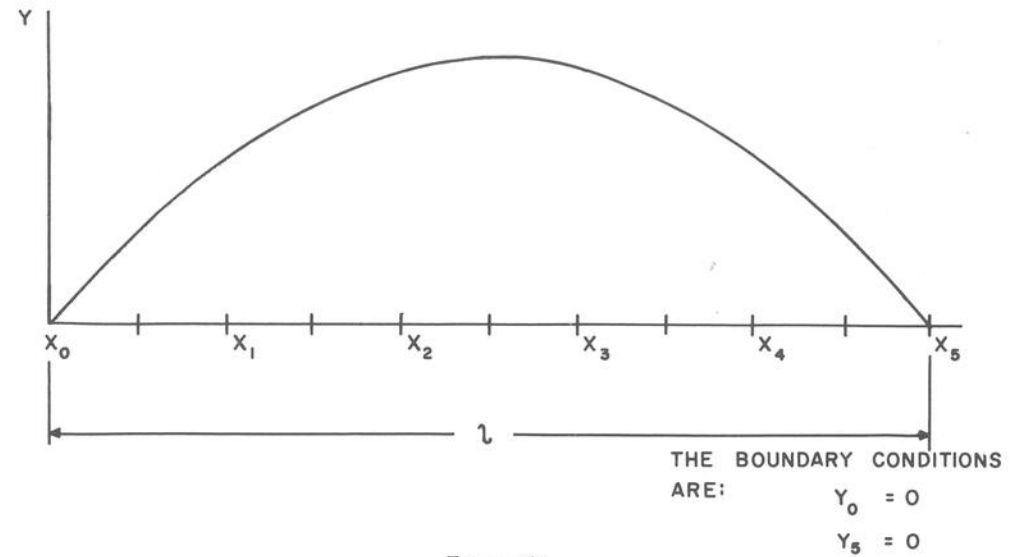
Figure 50.

$$\frac{d^2 y_4}{dt^2} = \frac{25 T}{\rho l^2}\left(-2y_4 + y_3\right) \tag{78}$$

Thus, by dividing the string into $n = 5$ segments we have changed the partial differential equation to $(n-1) = 4$ second order linear differential equations which can easily be programmed on the computer. The unscaled computer circuit is shown in *Figure 51*.

The above method for solving partial differential equations on the analog computer is the one most frequently used. Solutions are usually restricted to simple cases in two or possibly three dimensions. More information regarding computing methods for solving partial differential equations can be found in papers listed in the bibliography.

INTRODUCTION TO THE SOLUTION OF ALGEBRAIC EQUATIONS

Although the primary purpose of the analog computer is its use for the solution of differential equations, algebraic relations may have to be handled as well. One might wish to solve a set of simultaneous algebraic equations, and on the other hand, "algebraic loops" (circuits without integrators) can also appear in the computer circuit describing the dynamic behavior of a system. The necessary considerations of such algebraic relations are briefly described in the following.

Consider the set of algebraic equations:

$$a_{11}x_1 + a_{12}x_2 = b_1 \tag{79}$$

$$a_{21}x_1 + a_{22}x_2 = b_2 \tag{80}$$

where a_{11} and a_{22} are both positive coefficients. If we solve these equations using a similar approach as for differential equations, we obtain:

$$x_1 = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2 \tag{81}$$

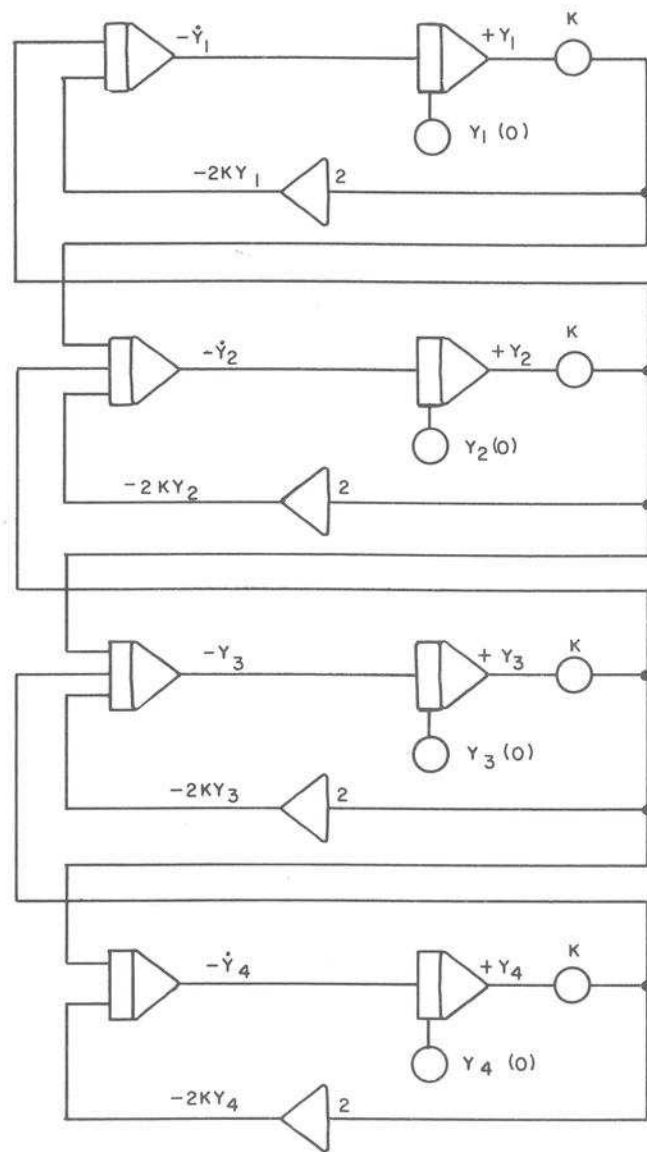$$x_2 = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1 \tag{82}$$

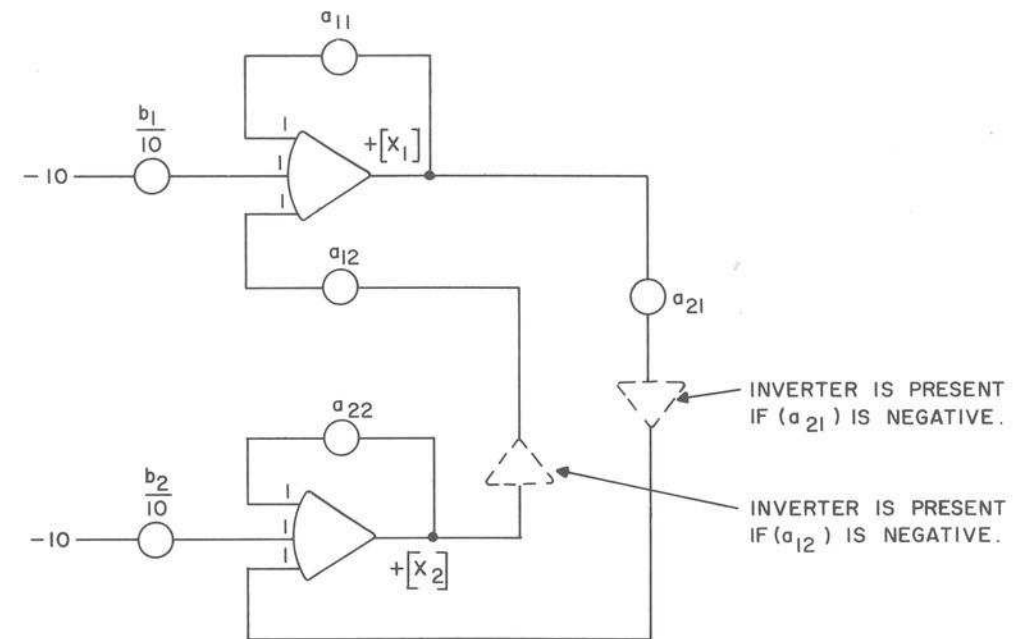$$K = \frac{25\,T}{\rho\,\ell\,2}$$

Figure 51.

Figure 52.

This requirement exists for all algebraic loops consisting of an even number of amplifiers. If it is not satisfied, an overloaded condition will quickly arise and no solution of the programmed problem will be obtained. Loops consisting of an odd number of summing amplifiers appear to be free of any difficulty of this nature. But unfortunately this is not true, for when no integrator is present in a loop with an odd number of amplifiers, phase shift unavoidably present at high frequencies within each amplifier will, for a given minimum loop gain, produce unwanted oscillations preventing stable conditions from occurring within the circuit. Thus, no matter the number of amplifiers, in a loop without an integrator one must be very careful that the loop gain is small if any problem solution is to obtained.

For these reasons a naive approach to the solution of algebraic equations is not guaranteed of success. A better approach for the setup of algebraic equations is to transform the set of algebraic equations into a set of linear differential equations having, as a steady state solution, the solution of the original algebraic equations. Again, considering the above example, we form the set of differential equations:

$$a_{11}x_1 + a_{12}x_2 - b_1 = -\frac{dx_1}{dt} \tag{84}$$

$$a_{21}x_1 + a_{22}x_2 - b_2 = -\frac{dx_2}{dt} \tag{85}$$

where $\frac{dx_i}{dt} = 0$ in steady state.

The computer circuit solving the equations is shown in *Figure 53*. This technique of solving simultaneous algebraic equations, which introduce integrators into the circuit is more acceptable because the integrators act as low pass filters which attenuate high frequency oscillations. A stable solution is assured if the inverter is present. However, the system may not reach steady state, if the product $a_{21}a_{12}$ is positive and

The computer circuit solving these equations is shown in *Figure 52*. The circuit might have only two amplifiers in the loop. A condition of positive rather than negative feedback then exists and the circuit will be electronically unstable unless the total loop gain K is less than unity.

$$K = \frac{(a_{12})\,(a_{21})}{(a_{11})\,(a_{22})} < 1 \tag{83}$$
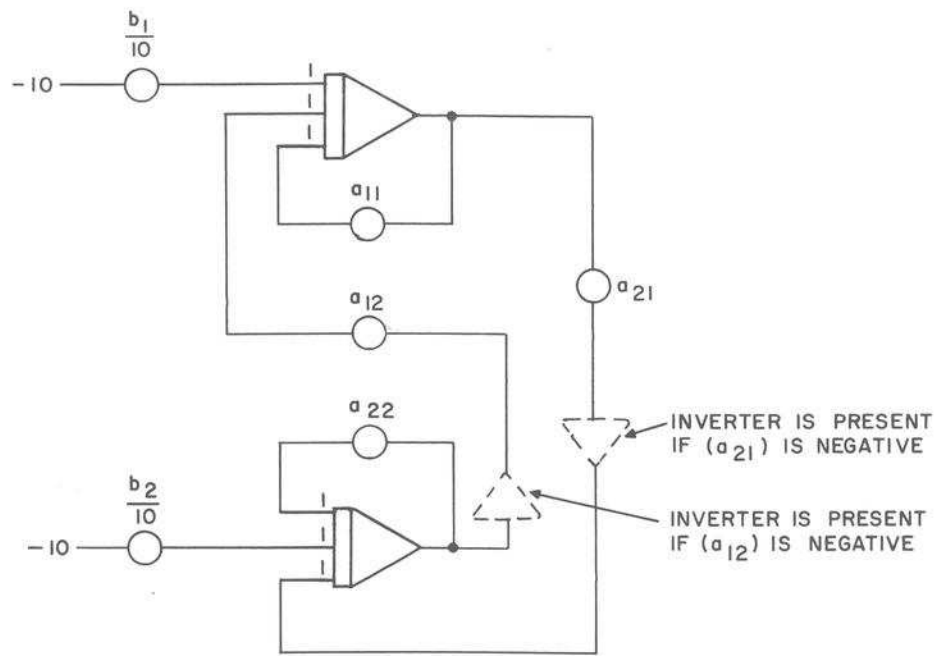
Figure 53.

no inverter is required. Thus, once more whether or not a solution is obtained depends on the particular problem and its programming. Several techniques for the guaranteed solution of algebraic equations with the analog computer are known and may be found in papers listed in the bibliography. One such technique, that of steepest ascents is briefly described in the following.

METHOD OF STEEPEST ASCENTS

This method can be employed when it is desired to find the values of x_i ($i = 1, 2, \ldots m$) so that the following relations are satisfied.

$$f_i(x_1, x_2, \ldots x_j \ldots x_m) = b_i \qquad i = 1 \ldots n \qquad (86)$$

where f_i are given real valued functions of the x_i and the b_i are constants, and $\dfrac{\partial f_i}{\partial x_j} \begin{cases} i = 1 \ldots n \\ j = 1 \ldots m \end{cases}$

are continuous. The computer solution is mechanized in such a fashion that the residual errors ϵ_i, defined as $f_i - b_i$ ($i = 1 \ldots n$), are minimized. In using a least square formulation for the error we define

$$s = \sum_{i=1}^{n} \epsilon_i^2 \qquad (87)$$

and

$$\frac{ds}{dt} = 2 \sum_{i=1}^{n} \epsilon_i \frac{d\epsilon_i}{dt} \qquad (88)$$

where

$$\frac{d\epsilon_i}{dt} = \sum_{j=1}^{m} \frac{\partial \epsilon_i}{\partial x_j} \frac{dx_j}{dt}, \ i = 1 \ldots n \qquad (89)$$
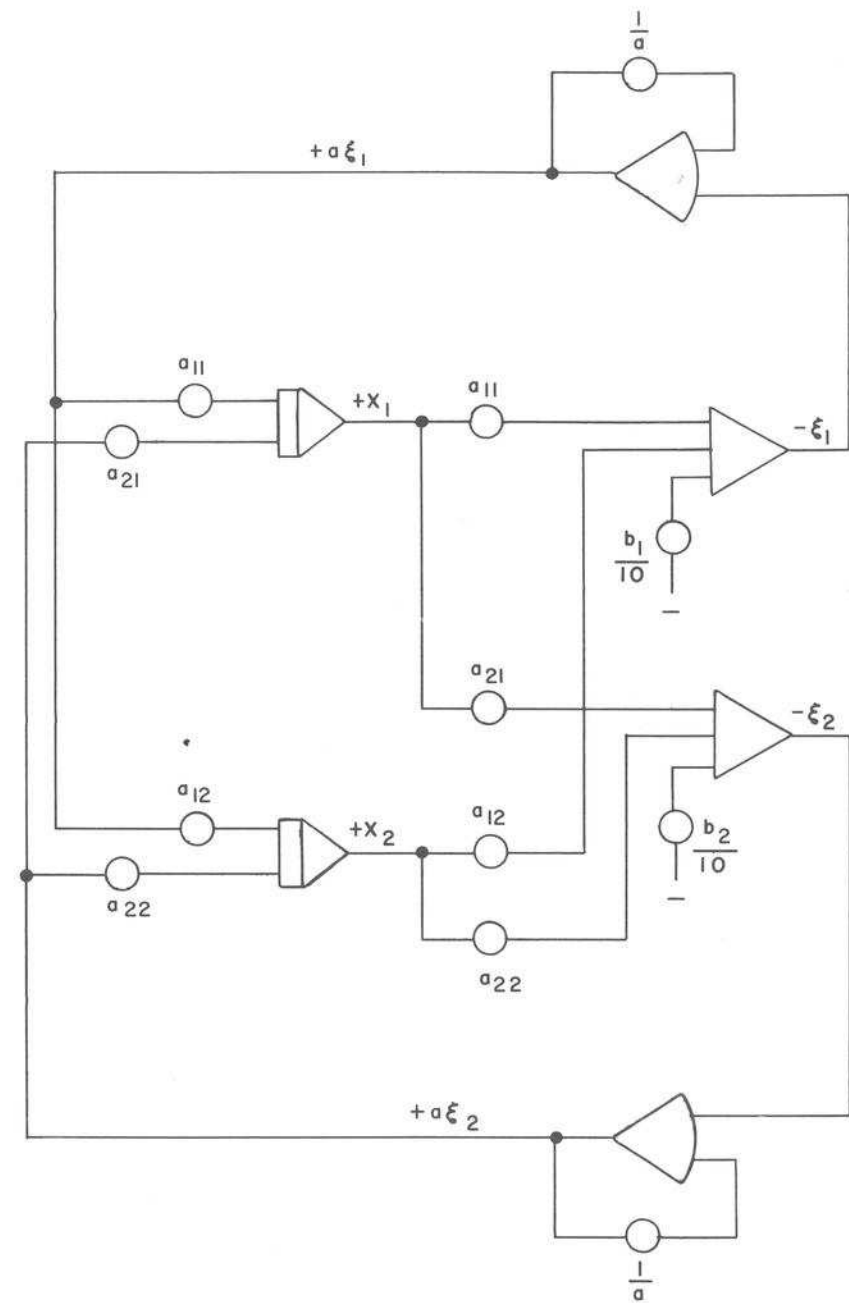
Figure 54.

From equations 88 and 89 one obtains:

$$\frac{ds}{dt} = 2 \sum_{i=1}^{n} \epsilon_i \sum_{j=1}^{m} \frac{\partial \epsilon_i}{\partial x_j} \frac{dx_j}{dt} \tag{90}$$

Interchanging summations, equation 90 becomes:

$$\frac{ds}{dt} = 2 \sum_{j=1}^{m} \frac{dx_j}{dt} \sum_{i=1}^{n} \epsilon_i \frac{\partial \epsilon_i}{\partial x_j} \tag{91}$$

The computer is now mechanized such that:

$$\frac{dx_j}{dt} = - a \sum_{i=1}^{n} \epsilon_i \frac{\partial \epsilon_i}{\partial x_j} \tag{92}$$

where a is a positive constant which determines the rate with which the solution converges. From equations 91 and 92 one obtains:

$$\frac{ds}{dt} = - \frac{2}{a} \sum_{j=1}^{m} \left(\frac{dx_j}{dt} \right)^2 \tag{93}$$

Note that the right side of equation 93 will always be negative, i.e. the computer continually tries to decrease s until $\frac{ds}{dt} = 0$. The values of x_j change until the minimum value of s is found. Where $m = n$, $\epsilon_i = 0$ and a unique solution for the x_j is obtained. If $m < n$, all n equations cannot be satisfied and then a solution is obtained for which s is a minimum. If $m < n$ more equations or constraints are necessary to determine a solution, for otherwise there are an infinite number of solutions.

A simple example will illustrate the use of the method of steepest ascents.

Assume a set of simultaneous algebraic equations of the form

$$a_{11}x_1 + a_{12}x_2 = b_1 \tag{94}$$

$$a_{21}x_1 + a_{22}x_2 = b_2 \tag{95}$$

Rewriting the equations for errors, yields:

$$a_{11}x_1 + a_{12}x_2 - b_1 = \epsilon_1 \tag{96}$$

$$a_{21}x_1 + a_{22}x_2 - b_2 = \epsilon_2 \tag{97}$$

With equation 92 we obtain:

$$\frac{dx_1}{dt} = - a \left[\epsilon_1 a_{11} + \epsilon_2 a_{21} \right] \tag{98}$$

$$\frac{dx_2}{dt} = - a \left[\epsilon_1 a_{12} + \epsilon_2 a_{22} \right] \tag{99}$$

Equations 96 through 99 can now be programmed on the computer. The resulting circuit is shown in *Figure 54*.

APPENDIX I. SYMBOLS FOR COMPUTER CIRCUITS

POTENTIOMETERS

a) GROUNDED

b) UNGROUNDED

OPERATIONAL AMPLIFIERS

a) INTEGRATOR

b) SUMMER

c) HIGH GAIN

SPECIAL FEEDBACK CONNECTIONS

X^2 DIODE FUNCTION GENERATOR TYPE 16.101
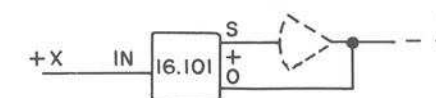
MULTIPLIER TYPE 7.045
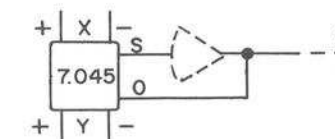
DUAL FUNCTION SWITCH TYPE 2.127
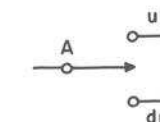
SIGNAL VOLTAGE COMPARATOR TYPE 6.143
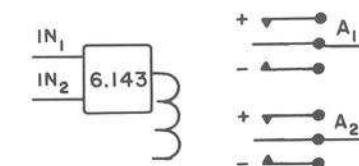
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